A New Look at Hamiltonian Walks

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ABSTRACT

Let G be a connected graph of order n. A Hamiltonian walk of G is a closed spanning walk of minimum length in G. For a cyclic ordering $s : v_1, v_2, \cdots, v_n, v_{n+1} = v_1$ of $V(G)$, let $d(s) = \sum_{i=1}^{n} d(v_i, v_{i+1})$, where $d(v_i, v_{i+1})$ is the distance between v_i and v_{i+1} for $1 \leq i \leq n$. Then the Hamiltonian number $h(G)$ of G is defined as $h(G) = \min\{d(s)\}$, where the minimum is taken over all cyclic orderings s of $V(G)$. It is shown that $h(G)$ is the length of a Hamiltonian walk in G. Thus $h(G) = n$ if and only if G is a Hamiltonian graph. It is also shown that $h(G) = 2n-2$ if and only if G is a tree. Moreover, for every pair n, k of integers with $3 \leq n \leq k \leq 2n-2$, there exists a connected graph G of order n having $h(G) = k$. The upper Hamiltonian number is defined as $h^+(G) = \max\{d(s)\}$, where the maximum is taken over all cyclic orderings s of $V(G)$. We show, for a connected graph G of order $n \geq 3$, that $h(G) = h^+(G)$ if and only if $G = K_n$ or $G = K_{1,n-1}$. We also study the upper Hamiltonian number of a tree and present bounds for the upper Hamiltonian number of a connected graph in terms of its order.

Key Words: Hamiltonian walk, Hamiltonian number.
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1 Introduction

In [6] Goodman and Hedetniemi introduced the concept of a *Hamiltonian walk* in a connected graph G, defined as a closed spanning walk of minimum length in G. They denoted the length of a Hamiltonian walk in G by $h(G)$. Therefore, for a connected graph G of order $n \geq 3$, it follows that $h(G) = n$ if and only if G is Hamiltonian. Among the results obtained by Goodman and Hedetniemi are the following.

Theorem A *If T is a tree of order n, then $h(T) = 2n - 2$.*

It is immediate that $h(G) \leq h(H)$ for each connected spanning subgraph H of a (connected) graph G. As a consequence of Theorem A, we can state the following.

Theorem B *For every connected graph G of order n,*

$$n \leq h(G) \leq 2n - 2.$$

Theorem C *If G is a k-connected graph of order n having diameter d, then*

$$h(G) \leq 2n - \left\lfloor \frac{k}{2} \right\rfloor (2d - 2) - 2.$$

Theorem D *Let G be a connected graph having blocks $B_1, B_2, \ldots, B_k$. Then the union of the edges in a Hamiltonian walk for each of the blocks B_i forms a Hamiltonian walk for G and, conversely, the edges in a Hamiltonian walk of G that belong to B_i form a Hamiltonian walk in B_i.*

Theorem D implies that the topic of Hamiltonian walks can be restricted to 2-connected graphs. Hamiltonian walks were studied further in [1, 2, 3, 5, 8, 9]. A well-known sufficient condition for a graph G to be Hamiltonian is due to Ore [7].

Theorem E *A graph G of order $n \geq 3$ is Hamiltonian if* $\deg u + \deg v \geq n$ *for every pair u, v of nonadjacent vertices of G.*

This theorem can be restated in terms of the parameter $h(G)$.

Theorem F *Let G be a graph of order $n \geq 3$. Then $h(G) = n$ if* $\deg u + \deg v \geq n$ *for every pair u, v of nonadjacent vertices of G.*

Bermond [3] obtained the following generalization of Theorem F.

Theorem G *Let G be a connected graph G of order $n \geq 3$ and let k be an integer with $0 \leq k \leq n - 2$. If* $\deg u + \deg v \geq n - k$ *every pair u, v of nonadjacent vertices of G, then $h(G) \leq n + k$.*

In this paper, we refer to the book [4] for graph theory notation and terminology not described here.

2 The Hamiltonian Number of a Graph

Of course, a Hamiltonian graph G contains a spanning cycle $C : v_1, v_2, \cdots, v_n, v_{n+1} = v_1$, where then $v_i v_{i+1} \in E(G)$ for $1 \le i \le n$. Thus Hamiltonian graphs of order $n \ge 3$ are those graphs for which there is a cyclic ordering $v_1, v_2, \cdots, v_n, v_{n+1} = v_1$ of $V(G)$ such that $\sum_{i=1}^{n} d(v_i, v_{i+1}) = n$, where $d(v_i, v_{i+1})$ is the distance between v_i and v_{i+1} for $1 \le i \le n$. For a connected graph G of order $n \ge 3$ and a cyclic ordering $s : v_1, v_2, \cdots, v_n, v_{n+1} = v_1$ of $V(G)$, we define the number $d(s)$ by

$$d(s) = \sum_{i=1}^{n} d(v_i, v_{i+1}).$$

Therefore, $d(s) \ge n$ for each cyclic ordering s of $V(G)$. The *Hamiltonian number* $h^*(G)$ of G is defined by

$$h^*(G) = \min\{d(s)\},$$

where the minimum is taken over all cyclic orderings s of $V(G)$. Consider the graph $G = K_{2,3}$ of Figure 1. For the cyclic orderings

$$s_1 : v_1, v_2, v_3, v_4, v_5, v_1 \text{ and } s_2 : v_1, v_3, v_2, v_4, v_5, v_1$$

of $V(G)$, we see that $d(s_1) = 8$ and $d(s_2) = 6$. Since G is a non-Hamiltonian graph of order 5 and $d(s_2) = 6$, it follows that $h^*(G) = 6$.

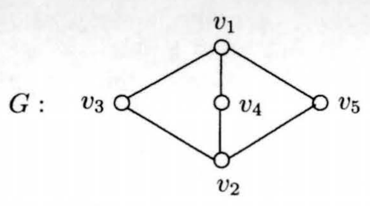

Figure 1: A graph G with $h^*(G) = 6$

We now see that there is an alternative way to define the length $h(G)$ of a Hamiltonian walk in G. Denote the length of a walk W by $L(W)$.

Proposition 2.1 *For every connected graph G,*

$$h^*(G) = h(G).$$

Proof. First, we show that $h(G) \leq h^*(G)$. Let $s: v_1, v_2, \cdots, v_n, v_{n+1} = v_1$ be a cyclic ordering of $V(G)$ for which $d(s) = h^*(G)$. For each integer i with $1 \leq i \leq n$, let P_i be a $v_i - v_{i+1}$ geodesic in G. Thus $L(P_i) = d(v_i, v_{i+1})$. The union of the paths P_i forms a closed spanning walk W in G. Therefore,

$$h(G) \leq L(W) = \sum_{i=1}^{n} L(P_i) = \sum_{i=1}^{n} d(v_i, v_{i+1}) = d(s) = h^*(G).$$

Next, we show that $h^*(G) \leq h(G)$. Let W be a Hamiltonian walk in G. Therefore, $L(W) = h(G)$. Suppose that $W: x_1, x_2, \ldots, x_N, x_1$, where then $N \geq n$. Define $v_1 = x_1$ and $v_2 = x_2$. For $3 \leq i \leq n$, define v_i to be x_{j_i}, where j_i is the smallest positive integer such that $x_{j_i} \notin \{v_1, v_2, \ldots, v_{i-1}\}$. Then $s: v_1, v_2, \ldots, v_n, v_{n+1} = v_1$ is a cyclic ordering of $V(G)$. For each i with $1 \leq i \leq n$, let W_i be the $v_i - v_{i+1}$ subwalk of W and so $d(v_i, v_{i+1}) \leq L(W_i)$. Since

$$h^*(G) \leq \sum_{i=1}^{n} d(v_i, v_{i+1}) \leq \sum_{i=1}^{n} L(W_i) = L(W) = h(G),$$

we have the desired result. ■

As a consequence of Proposition 2.1, we henceforth denote the Hamiltonian number of a graph G by $h(G)$, which is then the length of a Hamiltonian walk in G.

By Theorem A, if T is a tree of order n, then $h(T) = 2n - 2$. We now show that the converse of this statement holds as well. To do this, we first state a lemma.

Lemma 2.2 *If G is a connected graph such that $\delta(G) \geq 2$ and $\Delta(G) \geq 3$, then G contains two distinct cycles C and C' such that $V(C) \neq V(C')$.*

Theorem 2.3 *Let G be a connected graph of order n. Then $h(G) = 2n - 2$ if and only if G is a tree.*

Proof. By Theorem A, it suffices to show that if G is a connected graph of order $n \geq 3$ that is not a tree, then $h(G) < 2n - 2$. We proceed by induction on n. Since $h(K_3) = 3$, the result holds for $n = 3$. Suppose that $h(F) < 2(n-1) - 2 = 2n - 4$ for all connected graphs F of order $n - 1 \geq 3$ that are not trees. Let G be a connected graph of order $n \geq 4$ that is not a tree. Since $h(C_n) = n < 2n - 2$, we may assume that $G \neq C_n$.

We claim that G contains a vertex u such that $G - u$ is a connected subgraph of G that is not a tree. If G contains cut-vertices, then there is a vertex u that is a non-cut-vertex of an end-block that has the desired property. So we may assume that G is 2-connected and so $\delta(G) \geq 2$. By

Lemma 2.2, G contains two distinct cycles C and C' with $V(C) \neq V(C')$. Thus if u is a vertex that belongs to one of C and C' but not the other, then $G - u$ is a connected subgraph of G that is not a tree. By the induction hypothesis, $h(G - u) < 2(n-1) - 2 = 2n - 4$. Let

$$s_0 : v_1, v_2, \ldots, v_{n-1}, v_1$$

be a cyclic ordering of $V(G - u)$ with $d(s_0) = h(G - u) < 2n - 4$. Suppose that u is adjacent to the vertex v_i, where $1 \leq i \leq n - 1$. Define the cyclic ordering s_0' of $V(G)$ from s_0 by

$$s_0' : v_1, v_2, \ldots, v_i, u, v_{i+1}, \ldots, v_{n-1}, v_1.$$

Since $d(v_i, u) = 1$, it follows by the triangle inequality that

$$d(u, v_{i+1}) \leq 1 + d(v_i, v_{i+1}).$$

Therefore,

$$\begin{aligned} d(s_0') &= d(s_0) - d(v_i, v_{i+1}) + d(v_i, u) + d(u, v_{i+1}) \\ &\leq d(s_0) - d(v_i, v_{i+1}) + 1 + [1 + d(v_i, v_{i+1})] \\ &= d(s_0) + 2 < (2n - 4) + 2 = 2n - 2. \end{aligned}$$

Therefore, $h(G) \leq d(s_0') < 2n - 2$, as desired. ∎

By Theorem B, if G is a connected graph G of order n, then $n \leq h(G) \leq 2n-2$. Next we show that every pair k, n of integers with $3 \leq n \leq k \leq 2n-2$ is realizable as the Hamiltonian number and the order of some connected graph. In order to do this, we first present a known result, which is a consequence of Theorem D (see [6]).

Corollary H *Let G be a connected graph having blocks $B_1, B_2, \ldots, B_k$. Then*

$$h(G) = \sum_{i=1}^{k} h(B_i).$$

In particular, every bridge of G appears twice in every Hamiltonian walk of G.

Proposition 2.4 *For every pair n, k of integers with $3 \leq n \leq k \leq 2n-2$, there exists a connected graph G of order n having $h(G) = k$.*

Proof. For $k = n$, let G be a Hamiltonian graph of order n; while for $k = 2n - 2$, let G be a tree of order n. For $n < k < 2n - 2$, let $k = n + \ell$, where $1 \leq \ell \leq n - 3$. Now let G be the graph obtained from a cycle $C_{n-\ell} : u_1, u_2, \ldots, u_{n-\ell}, u_1$ and a path $P_\ell : v_1, v_2, \ldots, v_\ell$ by joining u_1 to v_1. Since $C_{n-\ell}$ is a block of G and any edge not on $C_{n-\ell}$ is a bridge of G, it then follows by Corollary H that

$$h(G) = h(C_{n-\ell}) + 2\ell = (n - \ell) + 2\ell = n + \ell = k,$$

as desired. ■

3 The Upper Hamiltonian Number of a Graph

We saw for the graph G of Figure 1 that there are cyclic orderings s_1 and s_2 of $V(G)$ such that $d(s_1) = 8$ and $d(s_2) = 6$. Indeed, it is not difficult to see that for *every* cyclic ordering s of $V(G)$, either $d(s) = 6$ or $d(s) = 8$.

For a connected graph G, we define the *upper Hamiltonian number* $h^+(G)$ by

$$h^+(G) = \max\{d(s)\},$$

where the maximum is taken over all cyclic orderings s of $V(G)$. From our remarks above, it follows that $h^+(K_{2,3}) = 8$. As an illustration, we now establish the upper Hamiltonian numbers of the hypercubes.

Proposition 3.1 *For each integer $n \geq 2$,*

$$h^+(Q_n) = 2^{n-1}(2n-1).$$

Proof. First, we show that $h^+(Q_n) \leq 2^{n-1}(2n-1)$. Let s be an arbitrary cyclic ordering of $V(Q_n)$ with $d(s) = h^+(Q_n)$. Since $\operatorname{diam} Q_n = n$ and for each vertex v in Q_n, there is exactly one vertex in Q_n whose distance from v is n, it follows that there are at most 2^{n-1} terms in $d(s)$ equal to n. Consequently, each of the remaining 2^{n-1} terms in $d(s)$ is at most $n-1$. Thus

$$d(s) \leq 2^{n-1}n + 2^{n-1}(n-1) = 2^{n-1}(2n-1),$$

and so $h^+(Q_n) \leq 2^{n-1}(2n-1)$.

Next we show that $h^+(Q_n) \geq 2^{n-1}(2n-1)$. Since the result is true for Q_2, we may assume that $n \geq 3$. Let $G = Q_n$. Then G consists of two disjoint copies G_1 and G_2 of Q_{n-1}, where corresponding vertices of G_1 and G_2 are adjacent. For each vertex v of G, there is a unique vertex $\overline{v}$ of G such that $d(v, \overline{v}) = n = \operatorname{diam} Q_n$. Necessarily, exactly one of v and $\overline{v}$ belongs to G_1 for each vertex v of G. It is well-known that Q_n is Hamiltonian for $n \geq 2$. Let $C : v_1, v_2, \ldots, v_{2^{n-1}}, v_{2^{n-1}+1} = v_1$ be a Hamiltonian cycle in G_1. Now define a cyclic ordering s of $V(G)$ by

$$s : v_1, \overline{v}_1, v_2, \overline{v}_2, \ldots, v_{2^{n-1}}, \overline{v}_{2^{n-1}}, v_1.$$

Since $d(v_i, \overline{v}_i) = n$ and $d(v_i, v_{i+1}) = 1$ for $1 \leq i \leq 2^{n-1}$, it follows by the triangle inequality that

$$n = d(v_i, \overline{v}_i) \leq d(v_i, v_{i+1}) + d(v_{i+1}, \overline{v}_i) = 1 + d(v_{i+1}, \overline{v}_i).$$

Thus $d(v_{i+1}, \overline{v}_i) \geq n-1$, which implies that $d(v_{i+1}, \overline{v}_i) = n-1$. Hence

$$h^+(Q_n) \geq d(s) = 2^{n-1}n + 2^{n-1}(n-1) = 2^{n-1}(2n-1),$$

as desired. ■

Obviously, $h(G) \leq h^+(G)$ for every connected graph G. For each integer $n \geq 3$, there are only two graphs G of order n for which $h(G) = h^+(G)$.

Theorem 3.2 *Let G be a connected graph of order $n \geq 3$. Then*

$$h(G) = h^+(G) \text{ if and only if } G = K_n \text{ or } G = K_{1,n-1}.$$

Proof. If $G = K_n$, then certainly $d(s) = n$ for every cyclic ordering s of $V(G)$; while if $G = K_{1,n-1}$, then $d(s) = 2n-2$ for every cyclic ordering s of $V(G)$. Thus $h(G) = h^+(G)$ if $G = K_n$ or $G = K_{1,n-1}$.

For the converse, suppose that G is a connected graph of order $n \geq 3$ such that $G \neq K_n, K_{1,n-1}$. We show that $h(G) \neq h^+(G)$. Let $\operatorname{diam} G = d$. Since $G \neq K_n$, it follows that $d \geq 2$. We consider two cases, according to whether $d \geq 3$ or $d = 2$.

Case 1. $d \geq 3$. Let v_1 and v_{d+1} be vertices of G such that $d(v_1, v_{d+1}) = d$ and let $P : v_1, v_2, \ldots, v_{d+1}$ be a $v_1 - v_{d+1}$ geodesic in G. Let $W = V(G) - V(P)$. If $W \neq \emptyset$, then let $W = \{w_1, w_2, \ldots, w_\ell\}$, where $\ell = n - d - 1$. Define a cyclic ordering s of $V(G)$ by

$$s : v_1, v_2, v_3, \ldots, v_{d+1}, v_1 \text{ or} \tag{1}$$

$$s : v_1, v_2, v_3, \ldots, v_{d+1}, w_1, w_2, \ldots, w_\ell, v_1, \tag{2}$$

according to whether $W = \emptyset$ or $W \neq \emptyset$. Let s' be the cyclic ordering of $V(G)$ obtained from s by interchanging the locations of v_2 and v_3 in s; that is,

$$s' : v_1, v_3, v_2, v_4, \ldots, v_{d+1}, v_1 \tag{3}$$

$$\text{or } s' : v_1, v_3, v_2, v_4, \ldots, v_{d+1}, w_1, w_2, \ldots, w_\ell, v_1, \tag{4}$$

according to whether $W = \emptyset$ or $W \neq \emptyset$. In either case, $d(s') = d(s) + 2$ and so $h(G) \neq h^+(G)$.

Case 2. $d = 2$. Since G is not a star, it follows that G is not a tree. Thus the girth $g(G) = k \geq 3$. Assume first that $k = 3$. Since G is connected and $G \neq K_n$, there exists a set U of four vertices of G such that $\langle U \rangle = K_4 - e$ or $\langle U \rangle$ is a triangle with a pendant edge. Therefore, we may assume, without loss of generality, that G contains one of the graphs F_1 and F_2 in Figure 2 as an induced subgraph. In either case, define the cyclic orderings s and s' as

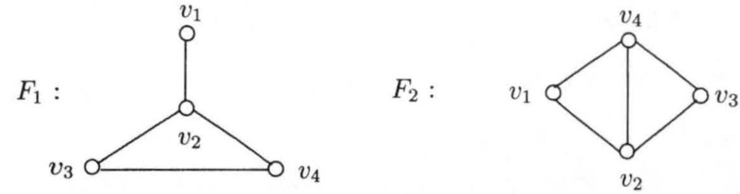

Figure 2: Induced subgraphs F_1 and F_2 of G

described in (1) (or (2)) and (3) (or (4)), respectively. Then $d(s') = d(s)+1$ and so $h(G) \neq h^+(G)$.

If $k \geq 4$, then let $C : v_1, v_2, \ldots, v_k, v_1$ be an induced cycle of G and let $V(G) - V(C) = \{w_1, w_2, \ldots, w_\ell\}$ if $\ell = n - k > 0$. Define the cyclic orderings s and s' of $V(G)$ as in (1) (or (2)) and (3) (or (4)), respectively. Since $d(s') = d(s) + 2$, it follows that $h(G) \neq h^+(G)$. ■

4 Bounds for the Upper Hamiltonian Number of a Graph

First, we observe that if $s : v_1, v_2, \ldots, v_n, v_{n+1} = v_1$ is any cyclic ordering of the vertex set of a connected graph, then for each vertex v_i ($1 \leq i \leq n$), both $d(v_{i-1}, v_i) \leq e(v_i)$ and $d(v_i, v_{i+1}) \leq e(v_i)$, where the subscripts are expressed as integers modulo n and $e(v_i)$ is the eccentricity of v_i (the distance from v_i to a vertex farthest from v_i). Thus, If G is a connected graph of order $n \geq 3$ and $V(G) = \{v_1, v_2, \ldots, v_n\}$, then

$$h^+(G) \leq \sum_{i=1}^{n} e(v_i).$$

Since the eccentricity of a vertex in G is at most the diameter $\operatorname{diam} G$ of G (the largest distance between two vertices of G), we have the following.

Proposition 4.1 *If G is a connected graph of order $n \geq 3$ and diameter d, then*

$$h^+(G) \leq nd.$$

The upper bound in Proposition 4.1 is sharp. For example, consider the odd cycle $C_{2k+1} : v_1, v_2, \ldots, v_{2k+1}, v_1$, where $k \geq 1$. Since $\operatorname{diam} C_{2k+1} = k$, it follows by Proposition 4.1 that $h^+(C_{2k+1}) \leq k(2k+1)$. On the other hand, let

$$s : v_1, v_{k+1}, v_{2k+1}, v_{3k+1}, \ldots, v_{(2k)k+1}, v_{(2k+1)k+1} = v_1,$$

where each subscript is expressed modulo $2k+1$ as one of the integers $1, 2, \ldots, 2k+1$. Since k and $2k+1$ are relatively prime, s is a cyclic ordering of $V(C_{2k+1})$. Since

$$d(s) = \sum_{i=1}^{n} d(v_i, v_{i+1}) = k(2k+1),$$

we have the following result.

Proposition 4.2 *For every integer $k \geq 1$, let $n = 2k+1$. Then $h^+(C_n) = nd$, where $d = \operatorname{diam} C_{2k+1}$.*

Therefore, the upper bound in Proposition 4.1 is attained for odd cycles. The situation for even cycles is far less clear. For every integer $k \geq 2$, we know that $h^+(C_{2k}) \geq 2k^2 - 2k + 2$. Indeed, we state the following.

Conjecture 4.3 *For every integer $k \geq 2$, $h^+(C_{2k}) = 2k^2 - 2k + 2$.*

Next, we study the upper Hamiltonian number of a tree. For each edge e of a tree T, we define the *component number* $\operatorname{cn}(e)$ of e as the minimum order of a component of $T - e$. For example, the edge e_3 of the tree T of Figure 3(a) has component number 3 since the order of the smaller component of $T - e_3$ is 3. Each edge of this tree is labeled with its component number in Figure 3(b).

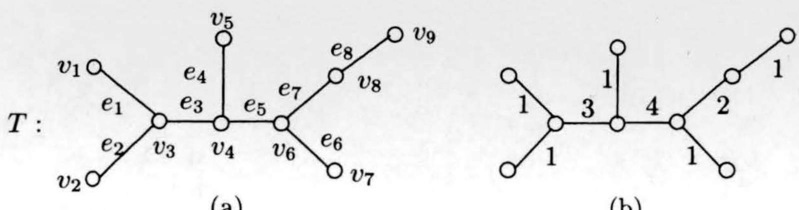

Figure 3: Component numbers of edges

We now present an upper bound for the upper Hamiltonian number of a tree.

Lemma 4.4 *Let T be a tree of order n with $E(T) = \{e_1, e_2, \ldots, e_{n-1}\}$. Then*

$$h^+(T) \leq 2 \sum_{i=1}^{n-1} \operatorname{cn}(e_i).$$

Proof. Let $e \in E(T)$, where T_1 and T_2 are the two components of $T - e$ and T_i has order n_i $(i = 1, 2)$. Assume, without loss of generality, that $n_1 \leq n_2$. Thus $\mathrm{cn}(e) = n_1$. Let s be a cyclic ordering of $V(T)$, say $s : v_1, v_2, \ldots, v_n, v_{n+1} = v_1$. For each i $(1 \leq i \leq n)$, the edge e occurs at most once in the $v_i - v_{i+1}$ path P_i of T. If e lies on P_i, then exactly one of v_i and v_{i+1} belongs to T_1. Since a vertex of T_1 can occur as the initial or terminal vertex of a path P_i $(1 \leq i \leq n)$ at most $2\,\mathrm{cn}(e)$ times, the desired result follows. ∎

For the tree T of Figure 3,

$$\sum_{i=1}^{8} \mathrm{cn}(e_i) = 1 + 1 + 3 + 1 + 4 + 1 + 2 + 1 = 14.$$

Thus by Lemma 4.4, $h^+(T) \leq 28$. However, for

$$s : v_1, v_9, v_2, v_8, v_3, v_7, v_5, v_6, v_4, v_1,$$

we have $d(s) = 28$. Therefore, $d(s) = 28 \leq h^+(T)$ and so $h^+(T) = 28$.

We now present a formula for $h^+(P_n)$.

Proposition 4.5 *For each $n \geq 2$,*

$$h^+(P_n) = \lfloor n^2/2 \rfloor.$$

Proof. Let $P_n : v_1, v_2, \ldots, v_n$ and let

$$s : v_1, v_n, v_2, v_{n-1}, v_3, \ldots, v_{\lceil \frac{n+1}{2} \rceil}, v_1.$$

Then

$$\begin{aligned} d(s) &= (n-1) + (n-2) + \ldots + 1 + \left\lceil \frac{n-1}{2} \right\rceil \\ &= \binom{n}{2} + \left\lceil \frac{n-1}{2} \right\rceil = \left\lceil \frac{n^2-1}{2} \right\rceil = \left\lfloor \frac{n^2}{2} \right\rfloor. \end{aligned}$$

Hence $h^+(P_n) \geq \left\lfloor \frac{n^2}{2} \right\rfloor$.

To show that $h^+(P_n) \leq \left\lfloor \frac{n^2}{2} \right\rfloor$, we consider two cases, according to whether n is odd or n is even. Let $e_i = v_i v_{i+1}$, $1 \leq i \leq n-1$.

Case 1. *n is odd, say $n = 2k + 1$, where $k \geq 1$.* Then

$$\mathrm{cn}(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq k \\ n - i & \text{if } k+1 \leq i \leq 2k. \end{cases}$$

By Lemma 4.4,

$$\begin{aligned} h^+(P_n) &\le 2 \sum_{i=1}^{n-1} \mathrm{cn}(e_i) = 2\left[\sum_{i=1}^{k} \mathrm{cn}(e_i) + \sum_{i=k+1}^{2k} \mathrm{cn}(e_i)\right] \\ &= 4\sum_{i=1}^{k} i = 4\binom{k+1}{2} = 4\binom{\frac{n+1}{2}}{2} = \frac{n^2-1}{2}. \end{aligned}$$

Case 2. *n is even, say* $n = 2k$, *where* $k \ge 1$. Then

$$\mathrm{cn}(e_i) = \begin{cases} i & \text{if } 1 \le i \le k \\ n-i & \text{if } k+1 \le i \le 2k-1. \end{cases}$$

By Lemma 4.4,

$$\begin{aligned} h^+(P_n) &\le 2 \sum_{i=1}^{n-1} \mathrm{cn}(e_i) = 2\left[\sum_{i=1}^{k} i + \sum_{i=k+1}^{2k-1} (n-i)\right] \\ &= 2\left[\sum_{i=1}^{k} i + \sum_{i=1}^{k-1} i\right] = 2\left[2\sum_{i=1}^{k-1} i + k\right] \\ &= 2\left[2\binom{k}{2} + k\right] = 4\binom{\frac{n}{2}}{2} + n = \frac{n^2}{2}. \end{aligned}$$

Thus, in each case, $h^+(P_n) \le \left\lfloor \frac{n^2}{2} \right\rfloor$, producing the desired result. ∎

If T is a tree of order n and T' is a tree obtained by adding a pendant edge to T, then $cn_T(e) \le cn_{T'}(e) \le cn_T(e) + 1$ for every edge e of T. We now show that the upper bound is attained for at most half of the edges of T. With the aid of this fact, we will be able to establish a sharp upper bound for the upper Hamiltonian number of a graph in terms of its order.

Lemma 4.6 *Let T be a tree of order n, and let T' be a tree obtained by adding a pendant edge to T. Then there are at most $(n-1)/2$ edges e in T such that* $\mathrm{cn}_{T'}(e) = \mathrm{cn}_T(e) + 1$.

Proof. For each $e \in E(T)$, let T_{1e} and T_{2e} be the two components of $T - e$ and let n_{1e} and n_{2e} be the orders of T_{1e} and T_{2e}, respectively. Assume, without loss of generality, that $n_{1e} \le n_{2e}$. Thus $cn(e) = n_{1e}$. Let $e_0 = xy$ be an edge of T such that $n_{2e_0} - n_{1e_0} \le n_{2e} - n_{1e}$ for all edges e in T. Suppose that T' is obtained from T by adding the pendant edge uv at the vertex u of T. We show that the number of edges e in T such that $\mathrm{cn}_{T'}(e) = \mathrm{cn}_T(e) + 1$ is at most $(n-1)/2$. Let T_1 and T_2 be the two components of $T - e_0$ such that $\mathrm{cn}(e_0)$ is the order of T_1 We may assume that $x \in V(T_1)$ and $y \in V(T_2)$. For each $e \in E(T)$, let T'_{1e} and T'_{2e} be the two components of $T' - e$ and let n'_{1e} and n'_{2e} be the orders of T'_{1e} and T'_{2e}, respectively. We may assume that $n'_{1e} \leq n'_{2e}$. We consider two cases.

Case 1. $u \in V(T_2)$. Let P be the $y - u$ path in T_2 (it is possible that $y = u$) as shown in Figure 4. Let $e \in E(T) - E(P)$. We consider two possibilities.

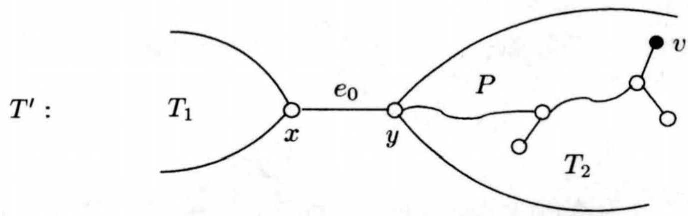

Figure 4: The tree T' in Case 1

Subcase 1.1. $e \in E(T_1) \cup \{e_0\}$. Then $T'_{1e} = T_{1e}$, while T'_{2e} is obtained by adding v and the edge uv to T_{2e}. Therefore, $\mathrm{cn}_{T'}(e) = n'_{1e} = n_{1e} = \mathrm{cn}_T(e)$ for all $e \in E(T_1) \cup \{e_0\}$.

Subcase 1.2. $e \in E(T_2) - E(P)$. We show that $\mathrm{cn}_{T'}(e) = \mathrm{cn}_T(e)$ in this subcase as well. Assume, to the contrary, that there exists $f \in E(T_2) - E(P)$ such that $\mathrm{cn}_{T'}(f) = \mathrm{cn}_T(f) + 1$. Then T'_{1f} is obtained by adding the pendant edge uv to T_{1f}, while $T'_{2f} = T_{2f}$. Since x and v are connected in $T' - f$ (by the path whose edge set is $E(P) \cup \{e_0, uv\}$) and $v \in E(T'_{1f})$, it follows that T_1 is a proper subgraph of T'_{1f}. Since T'_{1f} is obtained from T_{1f} by adding the pendant edge uv and $uv \notin E(T_1)$, it follows that T_1 is a proper subgraph of T_{1f} and so T_{2f} is a proper subgraph of T_2. This implies that $n_{1e_0} < n_{1f} \leq n_{2f} \leq n_{2e_0}$ and so $n_{2f} - n_{1f} < n_{2e_0} - n_{1e_0}$, which is impossible.

Therefore, if $e \in E(T)$ and $\mathrm{cn}_{T'}(e) = \mathrm{cn}_T(e) + 1$, then $e \in E(P)$. It remains to show that $|E(P)| \leq (n-1)/2$. Assume, to the contrary, that $|E(P)| \geq n/2$. Let

$$P' : y = v_0, v_1, \ldots, v_{|E(P)|} = u, v$$

be the path obtained by extending P to v. Let $f_0 = yv_1$ (see Figure 5). Then T_1 and the path $P' - y$ belong to different components in $T' - y$. Since the order of $P' - y$ is $|E(P)| + 1 \geq n/2 + 1$, it follows that $P' - y$ is a subgraph of T'_{2f_0}. Thus T_1 is a proper subgraph $T'_{1f_0} = T_{1f_0}$. Since T_{2f_0} is a subgraph T_2, it follows that $n_{1e_0} < n_{1f_0} \leq n_{2f_0} \leq n_{2e_0}$ and so $n_{2f_0} - n_{1f_0} < n_{2e_0} - n_{1e_0}$, which is impossible.

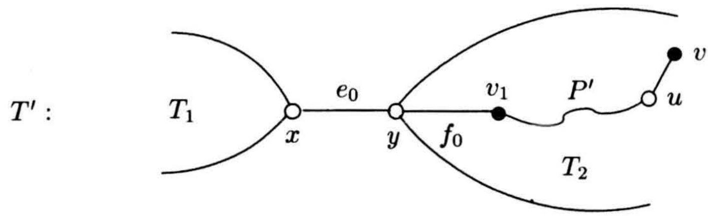

Figure 5: The path P' and the edge f_0 in T' in Case 1

Case 2. $u \in V(T_1)$. Let Q be the $u - x$ path in T_1 (it is possible that $u = x$) as shown in Figure 6. Let $e \in E(T) - (E(Q) \cup \{e_0\})$. We now consider two subcases.

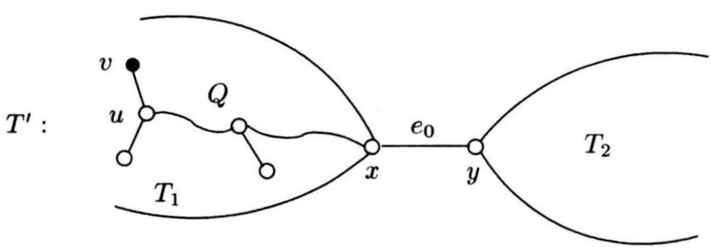

Figure 6: The tree T' in Case 2

Subcase 2.1. $e \in E(T_1) - E(Q)$. Then $T'_{1e} = T_{1e}$, while T'_{2e} is obtained by adding v and the edge uv to T_{2e}. Thus $\mathrm{cn}_{T'}(e) = \mathrm{cn}_T(e)$ for all $e \in E(T_1) - E(Q)$.

Subcase 2.2. $e \in E(T_2)$. We show that $\mathrm{cn}_{T'}(e) = \mathrm{cn}_T(e)$ in this subcase as well. Assume, to the contrary, that there exists $f \in E(T_2)$ such that $\mathrm{cn}_{T'}(f) = \mathrm{cn}_T(f) + 1$. Then T'_{1f} is obtained by adding v and the edge uv to to T_{1f}, while $T'_{2f} = T_{2f}$. Since v and y is connected in $T' - f$ (by the path whose edge set is $E(Q) \cup \{uv, e_0\}$) and $v \in V(T'_{1f})$, it follows that T_1 is a proper subgraph of T'_{1f}. Since T'_{1f} is obtained by adding v and the edge uv to to T_{1f} and $uv \notin E(T_1)$, it follows that T_1 is a proper subgraph of T_{1f} and so T_{2f} is a proper subgraph of T_2. This implies that $n_{1e_0} < n_{1f} \le n_{2f} < n_{2e_0}$ and so $n_{2f} - n_{1f} < n_{2e_0} - n_{1e_0}$, which is impossible.

Therefore, if $e \in E(T)$ and $\mathrm{cn}_{T'}(e) = \mathrm{cn}_T(e) + 1$, then $e \in E(Q) \cup \{e_0\}$. We now consider the vertex e_0. If $n_{1e_0} < n_{2e_0}$, then T'_{1e_0} is obtained from T_1 by adding the pendant edge uv and $T'_{2e_0} = T_2$, implying that $\mathrm{cn}_{T'}(e_0) = \mathrm{cn}_T(e_0) + 1$. If $n_{1e_0} = n_{2e_0}$, then T'_{2e_0} is obtained from T_1 by adding the pendant edge uv and $T'_{1e_0} = T_2$, implying that $\mathrm{cn}_{T'}(e_0) =$

$n_{2e_0} = n_{1e_0} = \mathrm{cn}_T(e_0)$. Thus, there are two possibilities.

Case i. $n_{1e_0} < n_{2e_0}$. Therefore, if $e \in E(T)$ and $\mathrm{cn}_{T'}(e) = \mathrm{cn}_T(e) + 1$, then $e \in E(Q) \cup \{e_0\}$. It remains to show that $|E(Q) \cup \{e_0\}| \leq (n-1)/2$ or $|E(Q)| \leq (n-3)/2$. If $|E(Q)| \geq (n-2)/2$, then the order of Q is at least $(n-2)/2 + 1 = n/2$. On the other hand, $n_{1e_0} < n_{2e_0}$ and so $n_{1e_0} < n/2$. However, Q is a subgraph of T_1, which is impossible.

Case ii. $n_{1e_0} = n_{2e_0}$. Therefore, if $e \in E(T)$ and $\mathrm{cn}_{T'}(e) = \mathrm{cn}_T(e) + 1$, then $e \in E(Q)$. It remains to show that $|E(Q)| \leq (n-1)/2$. If $|E(Q)| \geq n/2$, then the order of Q is at least $n/2 + 1$. However, Q is a subgraph of T_1 and the order of T_1 is at most $n/2$, which is impossible. ∎

An observation concerning trees that are not paths will also be useful.

Lemma 4.7 *If T is a tree of order $n \geq 5$ that is not a path, then there exists an end-vertex v in T such that $T - v$ is not a path.*

For trees that are not paths, we can now establish an upper bound for the sum of the component numbers of its edges.

Theorem 4.8 *If T is a tree of order $n \geq 4$ that is not a path, then*

$$2 \sum_{e \in E(T)} \mathrm{cn}(e) \leq \frac{n^2 - 4}{2}.$$

Proof. We proceed by induction on n. If $n = 4$, then $T = K_{1,3}$ is the only tree that is not a path. Since $h^+(K_{1,3}) = 6 = \frac{4^2}{2} - 2$, the result holds for $n = 4$. Suppose that the result holds for all trees of order $n - 1 \geq 4$ that are not paths. Let T be a tree of order $n \geq 5$ that is not a path. By Lemma 4.7 there exists an end-vertex v in T such that $T - v$ is not a path. Assume, without loss of generality, that $E(T) = \{e_1, e_2, \ldots, e_{n-2}, e_{n-1}\}$ and $E(T - v) = E(T) - \{e_{n-1}\}$. Then $\mathrm{cn}_T(e_{n-1}) = 1$ and by the induction hypothesis,

$$2 \sum_{i=1}^{n-2} \mathrm{cn}_{T-v}(e_i) \leq \frac{(n-1)^2 - 4}{2}. \quad (5)$$

It then follows by Lemma 4.6 that

$$\begin{aligned} 2 \sum_{i=1}^{n-1} \mathrm{cn}_T(e_i) &= 2 \sum_{i=1}^{n-2} \mathrm{cn}_T(e_i) + 2\mathrm{cn}_T(e_{n-1}) \\ &\leq 2 \left[\sum_{i=1}^{n-2} \mathrm{cn}_{T-v}(e_i) + \frac{n-2}{2} \right] + 2\mathrm{cn}_T(e_{n-1}) \end{aligned}$$

If n is even, then by (5)

$$\begin{aligned} 2\sum_{i=1}^{n-1} \mathrm{cn}_T(e_i) &\leq \frac{(n-1)^2-4}{2} + (n-2) + 2 \\ &\leq \frac{(n-1)^2-5}{2} + n = \frac{n^2-4}{2}. \end{aligned}$$

If n is odd, then by (5)

$$\begin{aligned} 2\sum_{i=1}^{n-1} \mathrm{cn}_T(e_i) &\leq 2\left[\sum_{i=1}^{n-2} \mathrm{cn}_{T-v}(e_i) + \frac{n-3}{2}\right] + 2 \\ &\leq \frac{(n-1)^2-4}{2} + (n-3) + 2 \leq \frac{n^2-5}{2}. \end{aligned}$$

Thus, in each case, $2\sum_{i=1}^{n-1} \mathrm{cn}_T(e_i) \leq \frac{n^2-4}{2}$, as desired. ∎

As with the Hamiltonian number, if G if a connected graph of order $n \geq 4$ and H is a connected spanning subgraph of G, then $h^+(G) \leq h^+(H)$. Thus, the following result follows by Theorem 4.8.

Corollary 4.9 *Let G be a connected graph of order $n \geq 4$ that is not a path. Then*

$$h^+(G) \leq \lfloor n^2/2 \rfloor - 2.$$

It then follows by Corollary 4.9 and Theorem 4.5 that there is no connected graph G of order $n \geq 4$ having $h^+(G) = \lfloor n^2/2 \rfloor - 1$. The following is a consequence of Theorems 2.3, 3.2, and 4.8.

Corollary 4.10 *Let T be a tree of order $n \geq 3$. Then*

$$2n - 2 \leq h^+(T) \leq \lfloor n^2/2 \rfloor.$$

Moreover,

(a) $h^+(T) = 2n-2$ *if and only if* $T = K_{1,n-1}$,

(b) $h^+(T) = \lfloor n^2/2 \rfloor$ *if and only if* $T = P_n$.

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