Observability of the Extended Lucas Cubes

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Abstract

A *Fibonacci string of order n* is a binary string of length n with no two consecutive ones. A Fibonacci string of order n which does not have a one in both the first and last postition is called a *Lucas string of order n*. The *Lucas cube* Λ_n is the subgraph of the hypercube Q_n induced by the set of Lucas strings. For positive integers i, n, with $n > i \geq 1$, the ith *extended Lucas cube* of order n, denoted by Λ_n^i, is a vertex induced subgraph of Q_n, where $V(\Lambda_n^i) = \hat{V}_n^i$ is defined recursively by the relation:

$$\hat{V}_n^i = \hat{V}_{n-1}^{i-1}0 + \hat{V}_{n-1}^{i-1}1$$

and the initial conditions $\hat{V}_1^0 = \{0, 1\}$, $\hat{V}_n^0 = V(\Lambda_n)$ for $n \geq 2$. We consider the number of colours required for a strong edge colouring of Λ_n^i and prove that for $n \geq 3$, $\text{obs}(\Lambda_n^i) = n + 1$ when $i = 1$ and $i = 2$, and obtain bounds on $\text{obs}(\Lambda_n^i)$ for $n > i \geq 3$.

Key words: Hypercube, Lucas cube, edge colouring, observability.

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1 Introduction

A proper edge colouring of a simple graph G is called *strong* if it is vertex distinguishing. The *observability* of G, denoted by obs(G), is the minimum number of colours required for a strong edge colouring of G. The parameter obs(G) was introduced by Černý et al. [2] and independently by Burris and Schelp [1], who denote it by $\chi'_s(G)$. In this study, we consider the observability of an infinite family of subgraphs of the hypercube (or n-cube) Q_n.

A *Fibonacci string of order n* is a binary string of length n with no two consecutive ones. A Fibonacci string of order n which does not have a one in both the first and last postition is called a *Lucas string of order n*.

Let V_n, $\hat{V}_n$ denote respectively the set of Fibonacci strings and the set of Lucas strings of order n. Let t, w be two binary strings. We denote by tw the string obtained by concatenating t and w. More generally, if S is any set of binary strings, let $tSw = \{tsw : s \in S\}$. Then for $n \geq 2$, the set V_n satisfies the recursive relation

$$V_n = 0V_{n-1} + 10V_{n-2}, \tag{1}$$

with initial conditions $V_0 = \{\emptyset\}$, $V_1 = \{0, 1\}$.

It is easily seen that $\hat{V}_1 = \{0\}$ and $\hat{V}_2 = \{00, 01, 10\}$. Then for $n \geq 3$, $\hat{V}_n$ is given by the recursive relation

$$\hat{V}_n = 0V_{n-1} + 10V_{n-3}0. \tag{2}$$

The *Fibonacci cube* Γ_n, proposed by Hsu [6], and the *Lucas cube* Λ_n, proposed by Munarini et al. [7], are the subgraphs of the hypercube Q_n induced respectively by the set of Fibonacci strings V_n and the set of Lucas strings $\hat{V}_n$. Thus the Lucas cubes are subgraphs of the Fibonacci cubes.

From relation (1), it can be seen that $|V_n|$ satisfies

$$|V_n| = |V_{n-1}| + |V_{n-2}| \tag{3}$$

and hence the sequence $|V_n|$ is a generalised Fibonacci sequence with initial terms $|V_0| = 1$ and $|V_1| = 2$. Recollect the Fibonacci sequence $\{F_n\}_{n=0}^{\infty} = 0, 1, 1, 2, 3, 5 \ldots$, so that $|V_n| = F_{n+2}$, $n \geq 0$.

In a similar way, it follows from (2) that for $n \geq 3$, $|\hat{V}_n|$ satisfies the recurrence relation

$$|\hat{V}_n| = |V_{n-1}| + |V_{n-3}| = F_{n+1} + F_{n-1}, \tag{4}$$

and hence when $n \geq 3$, $|\hat{V}_n|$ also satisfies the Fibonacci recurrence relation. Relation (4) gives $|\hat{V}_3| = 4$ and $|\hat{V}_4| = 7$ and hence the sequence

$\{|\hat{V}_n|\}_{n=1}^{\infty} = 1, 3, 4, 7, \ldots$ is a subsequence of the Lucas sequence $\{L_n\}_{n=0}^{\infty} = 2, 1, 3, 4, 7, \ldots$. In fact, we have $|\hat{V}_n| = L_n$, for $n \geq 1$.

The Fibonacci cubes have been generalised by Wu [10] to give an infinite family of subcubes of the hypercube. The Lucas cubes can also be generalised, in the following way. For positive integers i, n, with $n > i \geq 1$, the ith *extended Lucas cube* of order n, denoted by Λ_n^i, is a vertex induced subgraph of Q_n, where $V(\Lambda_n^i) = \hat{V}_n^i$ is defined recursively by the relation:

$$\hat{V}_n^i = \hat{V}_{n-1}^{i-1}0 + \hat{V}_{n-1}^{i-1}1 \tag{5}$$

and the initial conditions $\hat{V}_1^0 = \{0, 1\}$, $\hat{V}_n^0 = \hat{V}_n$ for $n \geq 2$. Thus it is easy to see that the vertices of Λ_n^i are (0,1)-strings of length n in which the last i positions are vertices of Q_i and the first $n - i$ positions are vertices of Λ_{n-i}. From definition (5), it follows immediately that for $n > i \geq 1$,

$$\Lambda_n^i = \Lambda_{n-1}^{i-1} \times K_2, \tag{6}$$

where $\Lambda_n^0 = \Lambda_n$ for $n \geq 2$ and $\hat{V}_1^0 = \{0, 1\}$. The construction of Λ_{n+1}^1 from Λ_n^0, for $n = 1, 2, 3$, is illustrated in Figure 1 below.

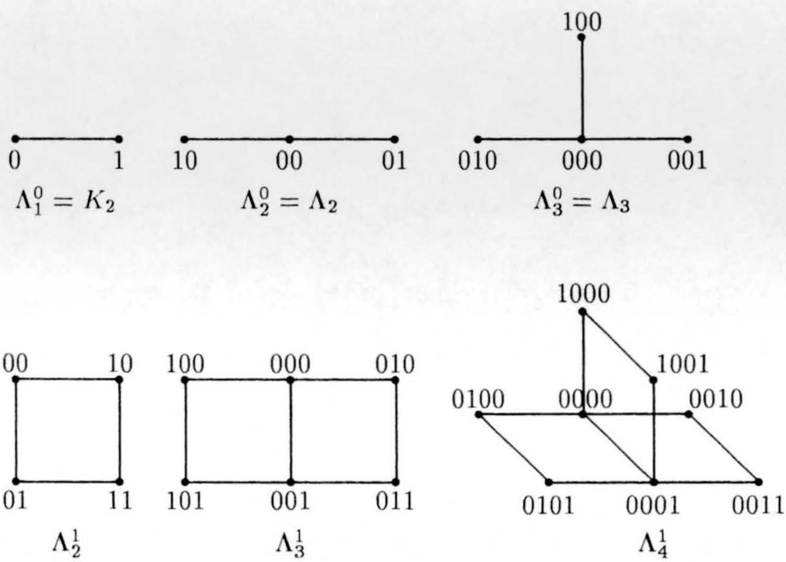

Figure 1.

By recursion from equation (6), we have

$$\Lambda_n^i = \Lambda_{n-j}^{i-j} \times Q_j, \tag{7}$$

for $2 \leq j \leq i < n$. Moreover, it is easy to see that $\Lambda_n^{n-1} \simeq Q_n$, for $n \geq 1$. In this study, we prove that for $n \geq 3$, $\text{obs}(\Lambda_n^i) = n+1$ when $i = 1$ and $i = 2$, and obtain bounds on $\text{obs}(\Lambda_n^i)$ for $i \geq 3$. An interesting property of the parameter obs is that if H is a proper subgraph of a graph G, then $\text{obs}(H)$ may be less than, equal to, or more than $\text{obs}(G)$. The value of $\text{obs}(Q_n)$ for low values of n and its assymptotic behaviour are established in [4], but for large values of n it is not known whether $\text{obs}(Q_n)$ is a strictly increasing function of n. It is therefore interesting to note (see [3]) that $\text{obs}(\Gamma_n) = \text{obs}(\Lambda_n)$ when $n \geq 4$, although Λ_n is a proper subgraph of Γ_n in this range. Moreover, the value of the observability of the extended Fibonacci cube Γ_n^i obtained in [8] implies that $\text{obs}(\Lambda_n^i) = \text{obs}(\Gamma_n^i)$, for $i = 1, 2$ and $n \geq 4$, although Λ_n^i is a proper subgraph of Γ_n^i for these values of i and n. The value of $\text{obs}(G)$ for some other special classes of graphs has been determined in [2] and [5]. Bounds on the value of $\text{obs}(G)$ in general are obtained in [1]. Structural properties of the Lucas cubes are discussed in [7] and of the extended Lucas cubes in [9].

2 Results

Noting that for all $n \geq 2$, the Lucas cube Λ_n contains a single vertex of maximum degree $\Delta(\Lambda_n) = n$, our first lemma is a direct deduction from equation (7).

Lemma 2.1 *Λ_n^i contains exactly 2^i vertices of degree $\Delta(\Lambda_n^i) = n$ when $1 \leq i \leq n-2$; and when $i = n-1$, all 2^n vertices have degree n.* □

We require the following results of Dedò et al. [3].

Lemma 2.2 (i) $\text{obs}(\Gamma_n) = n$, *for* $n \geq 4$;

(ii) $\text{obs}(\Lambda_n) = n$, *for* $n \geq 2$.

Theorem 2.3 *For* $n \geq 2$, $\text{obs}(\Lambda_{n+1}^1) = n+2$.

Proof. It follows from Lemma 2.1 that when $n \geq 2$, at least $n+2$ colours are required for a strong edge colouring of Λ_{n+1}^1. We shall prove that $n+2$ colours suffice. An example of a strong $(n+2)$-edge colouring of Λ_{n+1}^1 for $n = 2, 3, 4, 5, 6$ is shown in Figure A1, Figure A2 and Figure A3 in the Appendix.

Assume first that $n \geq 8$. By iterating a suitable number of times the decomposition (1) where $n \geq 2$, equation (2) gives

$$\begin{aligned} V_n &= 10010V_{n-6}0 + 00010V_{n-5} + 01010V_{n-5} + 1000V_{n-5}0 \\ &\quad + 0000V_{n-4} + 0100V_{n-4} + 1010V_{n-5}0 + 0010V_{n-4}. \end{aligned}$$

In the decomposition of Λ_n illustrated in Figure 2, each box represents the subgraph induced by the given sets of vertices. Thus, for example, the subgraphs generated by the vertex sets $00010(V_{n-6}0 + V_{n-7}01)$ and $1000(0V_{n-6} + 10V_{n-7})0$ are each isomorphic to Γ_{n-5}. Each bond between a pair of boxes represents a set of edges joining the vertices in the first box to their adjacent vertices in the second box.

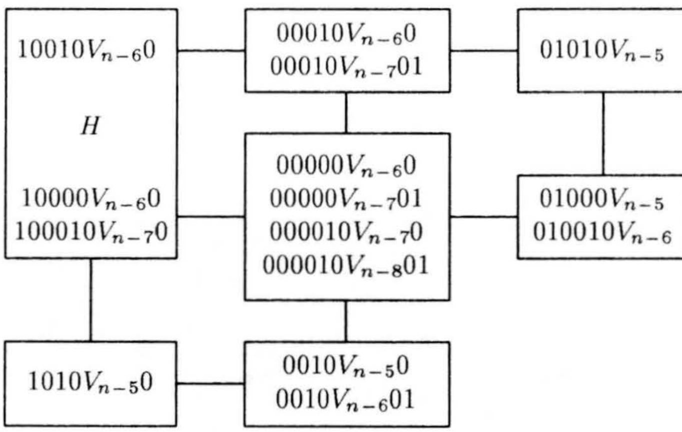

Figure 2.

Now consider the decomposition of Λ^1_{n+1} shown in Figure 3, given by taking two distinct copies of Λ_n, say G_1 and G_2, each decomposed as in Figure 2. In Figure 3, each of the subgraphs defined in Figure 2 (except for H) is represented by one of the vertices v_i in G_1 (w_i in G_2), $i = 2, 3, 5, 6, 7, 8$, and H is represented by the two vertices v_1, v_4 in G_1 (w_1, w_4 in G_2) joined by a double line. The vertices v_2, v_3 (w_2, w_3) represent subgraphs isormorphic to Γ_{n-5}, while v_5, v_6, v_7, v_8 (w_5, w_6, w_7, w_8) represent subgraphs isormorphic to Γ_{n-4}. The subgraph H induced by the vertices v_1, v_4 (w_1, w_4) is isomorphic to Γ_{n-4}.

We decompose in G_1 the set of vertices of the subgraph v_5 as $V(v_5) = S_{51} \cup S_{52} \cup S_{53} \cup S_{54}$, where $S_{51} = 00000V_{n-6}0$, $S_{52} = 00000V_{n-7}01$, $S_{53} = 000010V_{n-7}0$, $S_{54} = 000010V_{n-8}01$. Similarly, let $V(v_4) = S_{41} \cup S_{42}$, where

$S_{41} = 10000V_{n-6}0$, $S_{42} = 100010V_{n-7}0$. Let $S_{21} = 00010V_{n-6}0$, $S_{22} = 00010V_{n-7}01$, $S_{61} = 0100V_{n-5}0$, $S_{62} = 0100V_{n-6}01$. Then all the vertices of the subgraph v_5 are adjacent to vertices of v_6, v_8, w_5. In addition, the vertices of S_{51} are adjacent to vertices of S_{21}, S_{41}; vertices of S_{52} are adjacent to S_{22}; S_{53} to S_{42}. However, no vertex of S_{54} is adjacent to any vertex of v_2 or v_4. Moreover, the vertices $10010V_{n-6}0$ of the subgraph H are adjacent to vertices in S_{21}, but not to any vertex in S_{22}; vertices of v_7 are adjacent to vertices of $0010V_{n-5}0$ in v_8, but not to any vertex in $0010V_{n-6}01$; vertices of v_3 are adjacent to vertices in S_{61}, but not to any vertex in S_{62}. An analagous situation holds for the vertices of G_2.

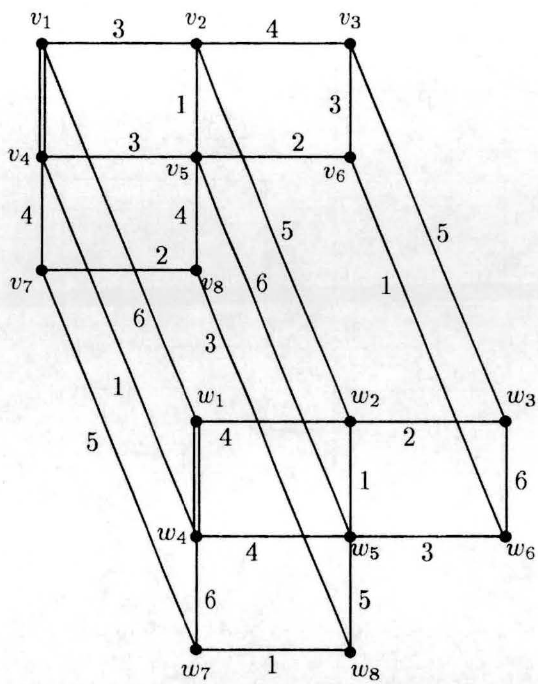

Figure 3.

In Figure 3, we have made the following assignment of colours: to the vertices of v_1, the set $\{3, 6\}$; to the vertices of v_2: $\{1, 3, 4, 5\}$, $\{1, 4, 5\}$; to the vertices of v_3: $\{3, 4, 5\}$; to the vertices of v_4: $\{1, 3, 4\}$; to the vertices of v_5: $\{1, 2, 3, 4, 6\}$, $\{1, 2, 4, 6\}$, $\{2, 3, 4, 6\}$, $\{2, 4, 6\}$; to the vertices of v_6: $\{1, 2, 3\}$, $\{1, 2\}$; to the vertices of v_7: $\{2, 4, 5\}$; to the vertices of v_8: $\{2, 3, 4\}$, $\{3, 4\}$. Similarly, to the vertices of w_1, we assign the set $\{4, 6\}$; to w_2: $\{1, 2, 4, 5\}$,

$\{1,2,5\}$; to w_3: $\{2,5,6\}$; to w_4: $\{1,4,6\}$; to w_5: $\{1,3,4,5,6\}$, $\{3,4,5,6\}$, $\{1,3,5,6\}$, $\{3,5,6\}$; to w_6: $\{1,3,6\}$, $\{1,3\}$; to w_7: $\{1,5,6\}$; to w_8: $\{1,3,5\}$, $\{3,5\}$. Note that all these colour sets are distinct and use just 6 colours.

Next consider the following edge colouring of Λ_8^1 obtained from the colouring shown in Figure 3 by assigning particular colours to the edges of the subgraphs denoted by v_i, w_i, $i = 2,3,5,6,7,8$, and both copies of H. In G_1 (G_2), the subgraphs v_5, v_6, v_8 (w_5, w_6, w_8) are each isomorphic to Γ_3 and have the colouring shown in Figure 4, where t is respectively $5, 4, y$ in v_5, v_6, v_8 ($1, 4, 6$ in w_5, w_6, w_8). H is also isomorphic to Γ_3 and both copies have the colouring shown in Figure 5. The subgraphs v_i, w_i, $i = 2,3,7$ are each isormorphic to $\Gamma_2 \cong \Lambda_2$ (see Figure 1) and in each of these subgraphs we give one edge the colour x and the other the colour y.

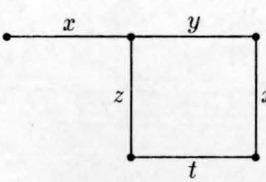

Figure 4.

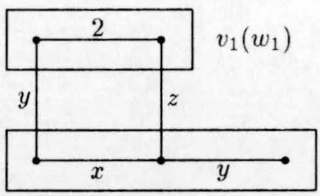

Figure 5.

For $i = 1, 2, \ldots 8$, this gives the following colour sets at the vertices of v_i:
$\{2,3,6,y\}$, $\{2,3,6,z\}$; $\{1,3,4,5,x\}$, $\{1,3,4,5,x,y\}$, $\{1,4,5,y\}$; $\{3,4,5,x\}$, $\{3,4,5,x,y\}$, $\{3,4,5,y\}$; $\{1,3,4,x,y\}$, $\{1,3,4,x,y,z\}$, $\{1,3,4,y\}$; $\{1,2,3,4,6,x\}$, $\{1,2,3,4,6,x,y,z\}$, $\{1,2,3,4,6,x,y\}$, $\{2,4,5,6,z\}$, $\{2,4,5,6,x\}$; $\{1,2,3,x\}$, $\{1,2,3,x,y,z\}$, $\{1,2,3,x,y\}$, $\{1,2,4,z\}$, $\{1,2,4,x\}$; $\{2,4,5,x\}$, $\{2,4,5,x,y\}$, $\{2,4,5,y\}$; $\{2,3,4,x\}$, $\{2,3,4,x,y,z\}$, $\{2,3,4,x,y\}$, $\{3,4,y,z\}$, $\{3,4,x,y\}$;
and at the vertices of w_i:
$\{2,4,6,y\}$, $\{2,4,6,z\}$; $\{1,2,4,5,x\}$, $\{1,2,4,5,x,y\}$, $\{1,2,5,y\}$; $\{2,5,6,x\}$, $\{2,5,6,x,y\}$, $\{2,5,6,y\}$; $\{1,4,6,x,y\}$, $\{1,4,6,x,y,z\}$, $\{1,4,6,y\}$; $\{1,3,4,5,6,x\}$, $\{1,3,4,5,6,x,y,z\}$, $\{1,3,4,5,6,x,y\}$, $\{1,3,5,6,z\}$, $\{1,3,5,6,x\}$; $\{1,3,6,x\}$, $\{1,3,6,x,y,z\}$, $\{1,3,6,x,y\}$, $\{1,3,4,z\}$, $\{1,3,4,x\}$; $\{1,5,6,x\}$, $\{1,5,6,x,y\}$, $\{1,5,6,y\}$; $\{1,3,5,x\}$, $\{1,3,5,x,y,z\}$, $\{1,3,5,x,y\}$, $\{3,5,6,z\}$, $\{3,5,6,x\}$.

These sets are distinct and hence we have a strong edge colouring of Λ_8^1 using 9 colours. Thus the theorem is also true when $n = 7$.

Now consider a strong edge colouring of Λ_{n+1}^1 when $n \geq 8$, obtained by assigning a strong edge colouring to each of the subgraphs isomorphic to Γ_j, $n-6 \leq j \leq n-4$, represented by the vertices v_i, w_i, $1 \leq i \leq 8$. Since $\text{obs}(\Gamma_n) \leq 4$ when $n \leq 4$, this can be done using at most $n-4$ distinct colours by Lemma 2.2 in the case when $n \geq 8$. We now assign to the edges in the decomposition of Λ^1_{n+1} represented by Figure 3 the colours shown in that figure. This gives a strong edge colouring of Λ^1_{n+1} using $n - 4 + 6 = n + 2$ colours. □

We require the following result proved in [8].

Lemma 2.4 *Let G be a graph containing at most one isolated vertex and no isolated edge. Then* $\text{obs}(G \times Q_2) \leq \text{obs}(G) + 3$. □

Theorem 2.5 $\text{obs}(\Lambda^2_n) = n + 1$, *for all* $n \geq 4$.

Proof. Since Λ^2_n contains at least two vertices of degree n for all $n \geq 4$, at least $n + 1$ colours are necessary for a strong edge colouring. However, by equation (7), $\Lambda^2_n = \Lambda_{n-2} \times Q_2$ and hence it follows from Lemma 2.4 and Lemma 2.2, that $n + 1$ colours also suffice. □

Theorem 2.6 *Let k be the least positive integer such that $\binom{k}{n} \geq 2^i$. Then for $i \geq 1$ and $n \geq i + 2$,*

$$k \leq \text{obs}(\Lambda^i_n) \leq n + \lceil i/2 \rceil.$$

Proof. The lower bound follows from Lemma 2.1 and cases $i = 1, 2$ are covered by Theorem 2.3 and Theorem 2.5. It therefore remains to establish the upper bound in the cases when $i \geq 3$. Consider first when $i = 2r$, $r \geq 2$. Then equation (7) gives $\Lambda^i_n = (\Lambda^2_{n-i+2} \times Q_{i-4}) \times Q_2$ and hence $\text{obs}(\Lambda^i_n) \leq \text{obs}(\Lambda^2_{n-i+2} \times Q_{i-4}) + 3$, by Lemma 2.4. Iterating, we obtain

$$\text{obs}(\Lambda^i_n) \leq \text{obs}(\Lambda^2_{n-i+2}) + 3(r-1) = n + r,$$

by Theorem 2.5.

A similar analysis in the case when $i = 2r + 1$, $r \geq 1$, using Theorem 2.3 gives

$$\text{obs}(\Lambda^i_n) \leq \text{obs}(\Lambda^1_{n-i+1}) + 3r = n + r + 1. \ \square$$

Corollary 2.7 $\text{obs}(\Lambda^3_n) = n + 2$ *when* $n = 5, 6$, *and* $\text{obs}(\Lambda^3_n) \leq n + 2$ *when* $n \geq 7$. □

Corollary 2.8 $\text{obs}(\Lambda^4_n) = n + 2$ *when* $6 \leq n \leq 14$, *and* $\text{obs}(\Lambda^4_n) \leq n + 2$ *when* $n \geq 15$. □

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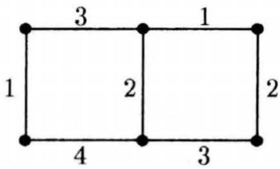
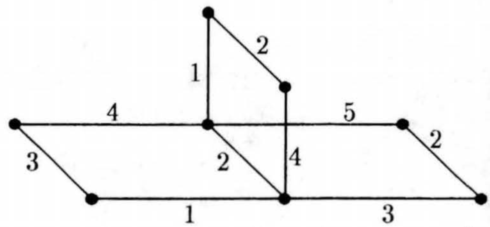
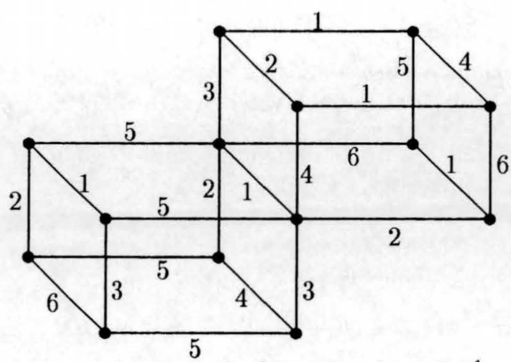
A Appendix



A strong 4-edge colouring of Λ_3^1

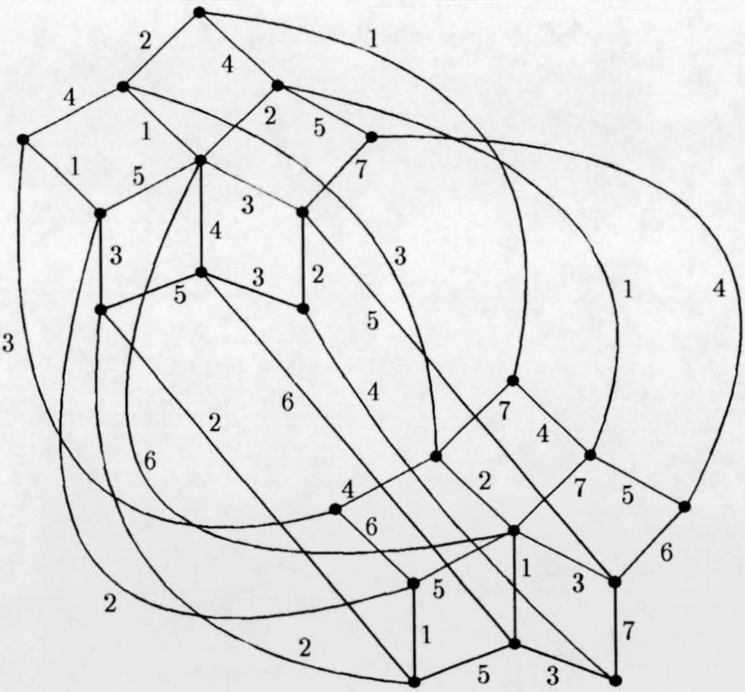


A strong 5-edge colouring of Λ_4^1



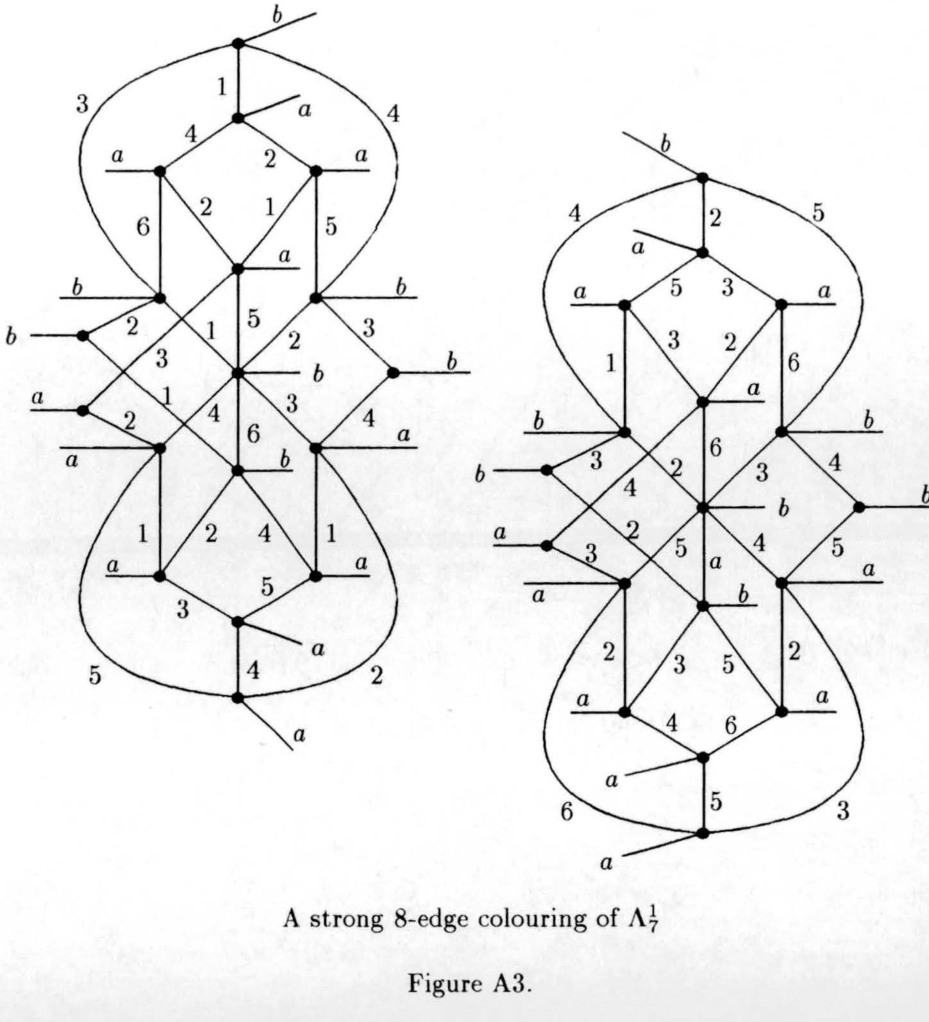
A strong 6-edge colouring of Λ_5^1

Figure A1.



A strong 7-edge colouring of Λ_6^1

Figure A2.



A strong 8-edge colouring of Λ_7^1

Figure A3.