# On the Nonexistence of Bent Hamiltonian Paths in the Grid Graph $P_3 \times P_5 \times P_5$

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**Abstract**

The existence of bent Hamiltonian paths in grid graphs $P_{n_1} \times P_{n_2} \times \cdots \times P_{n_d}$ has earlier been settled with the exception of the case $P_3 \times P_5 \times P_5$. Using a basic counting argument, that case is settled here.

A *grid graph* is a Cartesian product $P_{n_1} \times P_{n_2} \times \cdots \times P_{n_d}$, where $P_{n_i}$ is the path with $n_i$ vertices. W.l.o.g., we assume that $n_i \geq 2$. A *Hamiltonian path* in a graph $G$ is a sequence of adjacent vertices that contains each vertex of $G$ exactly once. A path in a grid graph that changes direction at every pair of successive edges is said to be *bent*.

Ruskey and Sawada [1] recently studied the existence of bent Hamiltonian paths in grid graphs $P_{n_1} \times P_{n_2} \times \cdots \times P_{n_d}$. They left the case $P_3 \times P_5 \times P_5$ open, and conjectured that there is no bent Hamiltonian path in that graph. We shall now show that this conjecture is true. The proof is based on a refinement of the counting argument in the proof of [1, Theorem 3].

**Theorem 1** *There is no bent Hamiltonian path in* $P_3 \times P_5 \times P_5$.

*Proof.* Let $G$ be the graph $P_3 \times P_5 \times P_5$ and label the vertices of $G$ with triples $(i, j, k)$, $0 \leq i \leq 2$, $0 \leq j, k \leq 4$. Assume that there is a bent

*Supported in part by the Academy of Finland under Grant No. 100500.

Hamiltonian path $P$ in $G$. We count the degrees in $P$ of the vertices $S := \{(i, j, k) : i, j, k \text{ even}\}$ in two ways.

Since $|S| = 2 \cdot 3 \cdot 3 = 18$ and all but at most two of the vertices have degree 2 in the path, the sum of the degrees is at least $2 \cdot 18 - 2 = 34$. All edges incident to a vertex in $S$ are along the paths where two of the coordinate values are fixed and even. There are 12 such paths with 5 vertices (and 4 edges) and 9 with 3 vertices (and 2 edges). As $P$ is bent, it can contain at most 2 and 1, respectively, of the edges of these paths, and each such edge contributes with 1 to the total degree. Hence the degree sum is at most $12 \cdot 2 + 9 \cdot 1 = 33$. Since $33 < 34$, we have a contradiction. □

Together with the results in [1], this completes the characterization of grid graphs that have a bent Hamiltonian path.

**Theorem 2** *There is a bent Hamiltonian path in* $P_{n_1} \times P_{n_2} \times \cdots \times P_{n_d}$ *iff (1)* $d = 1$ *and* $n_1 = 2$*; or (2)* $d = 2$ *and* $\min\{n_1, n_2\} = 2$*; or (3)* $d = 3$ *and the graph is (up to permuting the coordinates) neither* $P_3 \times P_3 \times P_{2k+1}$, $k \geq 1$ *nor* $P_3 \times P_5 \times P_5$*; or (4)* $d \geq 4$.

# References

[1] F. Ruskey and J. Sawada, Bent Hamilton cycles in $d$-dimensional grid graphs, *Electron. J. Combin.* **10** (2003), #R1.