Tournaments for Triads

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Abstract

We discuss some tournaments in which the underlying design has block size 3 and the order of elements is important.

1 Introduction

We consider a family of tournaments in which teams compete three at a time, and in which the order of teams is significant. We shall refer to the underlying 3-set in a match as a *triple*, the ordered triple as a *triad*, and the three positions in a match as *colors*.

Our interest in this problem arose from a request [2] for tournaments suitable for use in the paintball game *Zone*3. For that purpose, the following conditions were imposed:

(i) each round should contain the same number of matches;

(ii) every triple plays equally often in the tournament;

(iii) each team plays in each color equally often in the tournament;

(iv) each teach should play twice or three times per round;

(v) no team ever plays twice per round in the same color;

(vi) no two teams ever play each other twice in the same round;

(vii) the competition should last about 15 rounds.

A *good* tournament will mean one satisfying these conditions.

We shall refer to a set of triads in which every triple occurs once as an *iteration*; condition (ii) says that a good tournament consists of an integer number of iterations. Say there are n teams and t iterations. Then there are at most n matches per round, and consequently at least $\binom{n}{3}t/n = (n-1)(n-2)t/6$ rounds; even with $t = 1$, this rules out 12 or more teams (12 teams means at least 19 rounds). Each team competes in $3\binom{n}{3}t/n = (n-1)(n-2)t/2$ matches, so condition (iii) means that this number is divisible by 3; so $t = 3$ when 3 divides n. This rules out $n = 9$ (there would be at least 28 rounds). Condition (vi) clearly rules out $n < 5$, and $n = 5$ is also impossible (as $3 \nmid n$, at least one team plays three times in a round, and it needs six opponents). If $n = 6$, only the case where each team plays exactly twice need be considered.

If we interpret condition (vii) as meaning the number of rounds lies between 12 and 16, the parameters of good tournaments are given in Table 1. The "frequency" denotes the number of occurrences of teams in a round: 2^a3^b means that a teams compete twice and b teams compete once. Condition (i) implies that a and b will be constant within a design.

	n	m	F	F/I	t	TR
A	6	4	2^6	5	3	8
B	7	5	$2^6 3^1$	7	2	9
C	7	7	3^7	5	3	8
D	8	7	$2^3 3^5$	8	2	16
E	8	8	3^8	7	2	14
F	10	8	$2^6 3^4$	15	1	15
G	10	10	3^{10}	12	1	12
H	11	11	3^{11}	15	1	15

m = number of matches per round
F = frequency
R/I = number of rounds in one iteration
TR = total number of rounds

Table 1: Possible Types of Good Tournaments

We now construct examples of designs of these eight types. we shall use the following notation when working with the integers modulo v:

$$R(a,b,c) = \{(a+i, b+i, c+i) : i \in \mathbb{Z}_4\}.$$

2 Small examples

Type A. An example is shown in Table 2. The three triads derived from the triple $\{a, b, c\}$ have the form (a, b, c), (b, c, a) and (c, a, b) – they are even permutations of each other. Notice that the three iterations are intermingled – the design does not consist of three five-round designs, each, consisting of a single iteration satisfying (i) – (vi). (In fact no such five-round design exists.)

134	615	352	426	512	146	234	365	126	513	245	634
124	613	562	345	231	514	625	463	451	136	523	264
312	461	524	653	261	135	342	456	241	156	623	534
251	413	642	536	341	612	235	564	125	614	453	362
561	123	452	346	145	361	256	423	351	412	236	645

Table 2: Good Tournament, $n = 6$

Type B. Start with the round

127 316 451 532 674

and cycle modulo 7. This completes one iteration. Then perform a permutation of the colors, say (12), and repeat the whole iteration. This design has several interesting properties, which will be discussed in another paper.

Type C. The obvious method would be to find a design of five rounds which formed a single iteration, and repeat it twice with permutations. But this is impossible. Each round in such a design would constitute a Steiner triple system on seven points, and the set of five rows would be a large set of such systems. But no such large set exists (see [1]).

A design of type C can be constructed as follows. The teams are represented by A, B, 1, 2, 3, 4, 5, where the numbers are integers modulo 5 and A and B are invariants under addition modulo 5 ("infinity elements"). If the three rounds

123	514	$AB1$	$3A5$	$24A$	$B52$	$43B$
513	241	$1AB$	$43A$	$43A$	$B54$	$3B2$
134	251	$B1A$	$3A2$	$3A2$	$5B3$	$42B$

are developed modulo 5, the required design is found.

Type D. The first round is

123 245 364 487 538 672 751

and the other rounds are found by circulating modulo 8. Then another iteration is formed by property permuting the colors.

Type E. The first round is

$$147 \quad 264 \quad 723 \quad 435 \quad 651 \quad A12 \quad 3A6 \quad 57A$$

and the other rounds are found by circulating modulo 7, where $A + 1 = A \pmod 7$. Again, a second iteration is formed by taking a non-identity permutation of the colors.

Type F. One solution, with T representing 10, has first round

$$123 \quad 348 \quad 45T \quad 537 \quad 974 \quad 295 \quad 6T2 \quad 816,$$

rounds two to ten produced by circulating this round modulo 10, and the five rounds

$$\begin{array}{cccccccc} 235 & 346 & 457 & 679 & 78T & 891 & 9T2 & T13 \\ 125 & 347 & 458 & 569 & 67T & 892 & 9T3 & T14 \\ 128 & 239 & 451 & 562 & 673 & 784 & 9T6 & T17 \\ 129 & 23T & 341 & 563 & 674 & 785 & 896 & T18 \\ 124 & 568 & 236 & 781 & 895 & 34T & 452 & 9T7 \end{array}.$$

Type G. The first round is

$$123 \quad 348 \quad 45T \quad 537 \quad 974 \quad 295 \quad 6T2 \quad T69 \quad 816 \quad 781$$

where T again represents 10. The next nine rounds are formed by circulating modulo 10. The final two rounds are $R(1,2,4)$ and $R(1,2,9)$.

Type H. We write T, E for 10 and 11. Eleven rounds are formed from the round

$$123, \quad 45T, \quad 79E, \quad 369, \quad 148, \quad 572, \quad 46E, \quad 8T3, \quad 781, \quad E25, \quad 9T6$$

modulo 11. The other four rounds are

$$R(1,2,4), \quad R(1,2,5), \quad R(1,2,8) \quad \text{and} \quad R(1,2,T).$$

Although no good tournament for nine teams is possible, it is desirable to find a design with $n = 9$ which comes close to satisfying the conditions. Such a design is discussed in Section 3.

3 Designs for nine teams

Given nine objects, there is exactly one way (up to isomorphism) to select twelve triples such that every pair of objects occurs in exactly one triple. Such a selection is of course a *Steiner triple system* on 9 points; these have been widely discussed (see, for example, Chapter 12 of [5], or the relevant chapter in any elementary text discussing combinatorial designs). One example is the collection of sets

$$\begin{array}{ccc} 123, & 456, & 789, \\ 147, & 258, & 369, \\ 168, & 249, & 357. \end{array} \tag{1}$$

The rows are *resolutions* in the normal design-theoretic sense (every object occurs in exactly one block per row). This system can be represented by the array

$$\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 6 & 8 & 9 \end{array}$$

where the first three blocks are given by the rows of the array, the next three by the columns, the next three by the diagonals and the remainder by the back-diagonals.

We associate two structures with the triple system (1). The first is the set of nine triads

$$123, \quad 645, \quad 897, \quad 471, \quad 258, \quad 936, \quad 519, \quad 762, \quad 384 \tag{2}$$

which comprises a regular round. The second is the set

$$\begin{array}{cccccc} 123, & 645, & 897, & 471, & 258, & 936, \\ 159, & 267, & 348, & 816, & 924, & 735 \end{array} \tag{3}$$

consisting of two rounds in which no player competes twice in a round in the same color.

A *large set* of triple systems of order n is a partition of the set of all $\binom{n}{3}$ triples on n objects into $n-2$ triple systems. Such a set exists whenever $n \equiv 1$ or $3 \pmod 6$, except for $n = 7$ (see, for example, [3]). Table 3 shows a large set of triple systems of order 9, in array form (as given in [4]).

Suppose one applies construction (2) to each of these seven triple systems. Seven rounds are obtained, which satisfy conditions (i) and (iii) – (vi) of Section 2 and in which every triple is represented once or no times.

139	192	127	174	148	186	163
275	745	485	865	635	395	925
486	863	639	392	927	274	748

Table 3: A large set of triple systems of order 9

Two iterations provide a tournament of fourteen rounds. If (2) is replaced by

231, 645, 789, 174, 528, 963, 816, 493, 357

in the second iteration, and so on, the resulting fourteen-round tournament has the properties that every triple occurs once or twice and no triad is repeated.

Another design is found by taking the two rounds of (3) for each system. The resulting fourteen-round design has all properties except the overall balance property (iii).

References

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[3] L. Tierlinck, Large sets of disjoint designs and related structures, *Contemporary Design Theory - A Collection of Surveys* (Ed J. H. Dinitz and D. R. Stinson) (Wiley, New York, 1992), 561–592.

[4] R. A. Mathon, K. T. Phelps & A. Rosa, *Small Steiner triple systems and their properties*, *Ars Combin.* **15**(1983), 3–110.

[5] W. D. Wallis, *Combinatorial Designs* (Marcel Dekker, New York, 1988).