Two-fold Kirkman Packing Designs

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Abstract

This article investigates the spectrum of two-fold Kirkman Packing Designs and it is found that it contains all positive integers $v \geq 3$ except $5, 6$.

1 Introduction

Let X be a set of v points. A packing of X is a collection of subsets of X (called blocks) such that any pair of distinct points from X occur together in most λ block in the collection. A packing is called resolvable if its block set admits a partition into parallel classes, each parallel class being a partition of the point set X.

A Kirkman Triple System $KTS(v)$ is a collection $\mathcal{T}$ of 3-subsets of X (triples) such that any pair of distinct elements from X occur together in exactly one triple, and such that $\mathcal{T}$ admits a partition into $\frac{v-1}{2}$ parallel classes. Thus, a $KTS(v)$ is both a resolvable packing with $\lambda = 1$. It is well known that a $KTS(v)$ exists if and only if $v \equiv 3 \pmod 6$ (see, for example, [8]).

A Nearly Kirkman Triple System $NKTS(v)$ is a collection $\mathcal{T}$ of 3-subsets of X (triples) such that any pair of distinct elements from X occur together in at most one triple, and such that $\mathcal{T}$ admits a partition into $\frac{v}{2} - 1$ parallel classes. Thus an $NKTS(v)$ is both a resolvable packing with $\lambda = 1$. It is well known that an $NKTS(v)$ exists if and only if $v \equiv 0 \pmod 6$ and $v \geq 18$ (see, for example, [9]).

Černý, Horák and Wallis [3] introduced a particular generalization of Kirkman Triple System and Nearly Kirkman Triple System to the case where v is not a multiple of 3. They require all blocks to be of size three except that, each resolution class should contain either one block of size two (when $v \equiv 2 \pmod 3$) or one block of size four (when $v \equiv 1 \pmod 3$). They define a Kirkman packing design $KPD(v)$ to be a resolvable packing of a v-set by the maximum possible number of resolution classes of this type.

Some simple computation shows:

- a $KPD(v)$ contains at most $\frac{v}{2}$ resolution classes (when $v \equiv 2 \pmod 6$) or $\frac{v-1}{2}$ resolution classes (when $v \equiv 5 \pmod 6$).
- a $KPD(v)$ contains at most $\frac{v-3}{2}$ resolution classes (when $v \equiv 1 \pmod 6$) or $\frac{v}{2} - 2$ resolution classes (when $v \equiv 4 \pmod 6$).

Kirkman packing design have been studied by many researchers (see, for example, [3], [5] and [7]), the result was updated by Cao and the author recently.

Theorem 1.1 ([2]) *There is a $KPD(v)$ for the following cases:*

1. $v \equiv 2 \pmod 6$,
2. $v \equiv 5 \pmod 6$ *with* $v \geq 17$,
3. $v \equiv 4 \pmod 6$ *with* $v \geq 16$, *and*
4. $v \equiv 1 \pmod 6$ *with* $v \geq 19$.

In this article, we shall be restricting our attention to the case $\lambda = 2$. A two-fold Kirkman Triple System $KTS_2(v)$ is a collection $\mathcal{T}$ of 3-subsets of X (triples) such that any pair of distinct elements from X occur together in exactly two triple, and such that $\mathcal{T}$ admits a partition into $v-1$ parallel classes. Thus, a $KTS_2(v)$ is both a resolvable packing with $\lambda = 2$. It is well known that a $KTS_2(v)$ exists if and only if $v \equiv 0 \pmod 3$ and $v \neq 6$.

Theorem 1.2 ([6]) *There is a $KTS_2(v)$ if and only if $v \equiv 0 \pmod 3$ and $v \neq 6$.*

The problem we now study in this article is the two-fold Kirkman Packing Designs analogous of the Čerńy, Horák and Wallis [3]. We introduced the two-fold resolvable packing which requires all blocks to be of size three except that, each resolution class should contain either one block of size two (when $v \equiv 2 \pmod 3$) or one block of size four (when $v \equiv 1 \pmod 3$). We define a two-fold Kirkman packing design $KPD_2(v)$ to be a resolvable packing of a v-set by the maximum possible number of resolution classes of this type.

Some simple computation shows:

- a $KPD_2(v)$ contains at most v resolution classes (when $v \equiv 2 \pmod 3$).

- a $KPD_2(v)$ contains at most $v - 3$ resolution classes (when $v \equiv 1 \pmod 3$).

Take a two-fold Kirkman Triple System on $v+1$ points $KTS_2(v+1)$ and delete one point, we can dispense with the case $v \equiv 2 \pmod 3$ relatively quickly.

Theorem 1.3 *There is a* $KPD_2(v)$ *for every* $v \equiv 2 \pmod 3$ *except for* $v = 5$.

Proof. We only need prove there is no $KPD_2(5)$. Suppose there exists a $KPD_2(5)$, then there exists a parallel class of blocks, say $\{1,3\}$, $\{0,2,4\}$. The total number of parallel classes is 5, and accordingly, there are 4 parallel classes in addition to the mentioned parallel class. Each such parallel class contains one block of size two and one block of size three. If the pair of elements 1 and 3 is a leave or in a block of size two. Notice that each block of size three in the remaining 4 parallel classes must contain a pair of even elements, but there are only 3 such pairs remained. Consequently the construction is impossible. If the pair of elements 1 and 3 be in a block of size three. Notice that the block of size two in the parallel class must contain a pair of even elements and then each block of size three in the remaining 3 parallel classes must contain a pair of even elements, but there are only 2 such pairs remained. Consequently the construction is impossible.

In the remainder of this article we shall investigate the existence of $KPD_2(v)$ for every $v \equiv 1 \pmod 3$, and it is found that it contains all positive integers $v \equiv 1 \pmod 3$. That is, we will prove

Theorem 1.4 *There is a* $KPD_2(v)$ *for every* $v \equiv 1 \pmod 3$.

2 Preliminaries

In this section we shall define some of the auxiliary designs and some of the fundamental results which will be used later. The reader is refered to [4] for more information on designs, and, in particular, group divisible designs and frames.

Let K and M be sets of positive integers. A group divisible design (GDD) $GD(K, \lambda, M; v)$ is a triple $(X, \mathcal{G}, \mathcal{B})$ where

1. X is a v-set (of points),
2. $\mathcal{G}$ is a collection of nonempty subsets of X (called groups) with cardinality in M and which partition X,

3. $\mathcal{B}$ is a collection of subsets of X (called blocks) with cardinality at least two in K,

4. no block intersects any group in more than one point,

5. each pair set $\{x, y\}$ of points not contained in a group is contained in exactly λ blocks.

The group-type (or type) of the GDD $(X, \mathcal{G}, \mathcal{B})$ is the multiset of sizes $|G|$ of the $G \in \mathcal{G}$ and we usually use the "exponential" notation for its description: group-type $1^i 2^j 3^k \cdots$ denotes i occurences of groups of size 1, j occurences of groups of size 2, and so on.

A GDD $(K, \lambda, M; v)$ is resolvable if the blocks of $\mathcal{B}$ can be partitioned into parallel classes.

We need to establish some more notations. We shall denote by $GD(k, \lambda, m; v)$ a $GD(\{k\}, \lambda, \{m\}; v)$. If $m \notin M$, the $GD(K, \lambda, M \cup \{m*\}; v)$ denotes a $GD(K, \lambda, M \cup \{m\}; v)$ which contains a unique group of size m and if $m \in M$, then a $GD(K, \lambda, M \cup \{m*\}; v)$ is a $GD(K, \lambda, M; v)$ containing at least one group of size m. We shall sometimes refer to a GDD $GD(K, 1, M; v)$, $(X, \mathcal{G}, \mathcal{B})$ as a K-GDD. A transversal design $TD(k, n)$ is a $\{k\}$-GDD of type n^k. It is well known that a $TD(k, n)$ is equivalent to $k - 2$ mutually orthogonal Latin squares of order n.

A GDD $(X, \mathcal{G}, \mathcal{B})$ is called frame resolvable if its block set $\mathcal{B}$ admits a partition into holey parallel classes, each holey parallel class being a partition of $X - G_j$ for some $G_j \in \mathcal{G}$. A Kirkman Frame is a frame resolvable GDD in which all the blocks have size three. It is a simple consequence of the define that to each group G_j in a Kirkman Frame $(X, \mathcal{G}, \mathcal{B})$ there correspond exactly $\frac{\lambda |G_j|}{2}$ holey parallel classes of triples that partition $X - G_j$. The groups in a Kirman Frame are often referred to as holes.

For the two-fold Kirkman Frame we have

Theorem 2.1 ([1]) *A two-fold Kirkman Frame of type g^u exists if and only if $v \geq 4$ and $g(u-1) \equiv 0 \pmod 3$.*

We now illustrate the main technique that we will be using throughout the remainder of the article, which is a variant of Stinson's "Filling in Holes" construction. In applying the "Filling in Holes" construction, we will require two-fold Kirkman Frames in which the blocks are not necessarily all of the same size. To get these, we use the following "Weighting Construction".

Theorem 2.2 ([10]) *Suppose that there is a K-GDD of type $g_1^{t_1} g_2^{t_2} \cdots g_m^{t_m}$ and that for each $k \in K$ there is a two-fold Kirkman Frame of type h^k. Then there is a two-fold Kirkman Frame of type $(hg_1)^{t_1} (hg_2)^{t_2} \cdots (hg_m)^{t_m}$.*

Finally, as the "Filling in Holes" construction will generally involve adjoining more than one infinite point to a Kirkman Frame, we will require the notation of an incomplete two-fold Kirkman Packing Design. Let $v \equiv w \equiv 1 \pmod 3$, an incomplete two-fold Kirkman Packing Design, $IKPD_2(v,w)$, is a triple $(X, Y, \mathcal{B})$ where X is a set of v elements, Y is a subset of X of size w (Y is called the hole) and $\mathcal{B}$ is a collection of subsets of X (blocks), each of size 3 or 4, such that

1. $|Y \cap B_i| \leq 1$ for all $b_i \in \mathcal{B}$,

2. any pair of distinct elements in X occur together either in Y or in at most two blocks,

3. $\mathcal{B}$ admits a partition $v-w$ parallel classes on X, each of which contains one block of size four, and a further $w-3$ holey parallel classes of triple on $X \setminus Y$.

4. each element of $X \setminus Y$ is contained in exactly four blocks of size four.

Example 2.3 There is an $IKPD_2(v, 4)$ for $v \in \{16, 19, 22\}$.

Point Set: $X = Z_{v-4} \cup Y$, $Y = \{a, x_1, x_2, x_3\}$.
Parallel Classes: Develop the following class mod $(v-4)$:

$v = 16:$ $\{0,1,2,6\}, \{a,3,5\}, \{x_1,4,9\}, \{x_2,7,10\}, \{x_3,8,11\}$.
Holey Parallel Class: $\{i, i+4, i+8\} (0 \leq i \leq 3)$.

$v = 19:$ $\{0,1,4,11\}, \{3,6,12\}, \{a,2,9\}$,
$\{x_1,5,7\}, \{x_2,8,10\}, \{x_3,13,14\}$.
Holey Parallel Class: $\{i, i+5, i+10\} (0 \leq i \leq 4)$.

$v = 22:$ $\{0,1,2,4\}, \{3,6,13\}, \{7,11,16\}, \{a,5,12\}$,
$\{x_1,8,14\}, \{x_2,9,17\}, \{x_3,10,15\}$.
Holey Parallel Class: $\{i, i+6, i+12\} (0 \leq i \leq 5)$.

3 The main result

Lemma 3.1 *There exists a $KPD_2(v)$ for every $v \equiv 1 \pmod 3$ with $v \leq 49$, and for $v \in \{6t+1 : t \geq 9\}$.*

Proof. We construct directly $KPD_2(7)$ as follows:

$$\{0,1,2,3\}, \{4,5,6\}; \{0,1,4,5\}, \{2,3,6\};$$
$$\{0,2,4,6\}, \{1,3,5\}; \{0,3,5,6\}, \{1,2,4\}.$$

For the case $v \in \{6t+1 : t \geq 3\}$, we start with the design $KPD(v)$ and take two copies of each block to obtain the desired design. For the cases $v = 16$ and 22, we start with the design $IKPD(v,4)$ and fill the hole with a block of size 4 to obtain the desired design. For the case $v = 10$, see [11]. For the others see Appendix.

Lemma 3.2 *Suppose*

1. *there is a two-fold Kirkman Frame of type* $g_1 g_2 \cdots g_m$,
2. *there is an* $IKPD_2(g_i + w, w)$ *for every* $i < m$,
3. *there is a* $KPD_2(g_m + w)$.

Then there is a $KPD_2(\sum_{1\leq i\leq m} g_i + w)$.

Proof. We start with a two-fold Kirkman Frame of type $g_1 g_2 \cdots g_m$ $(X, \mathcal{G}, \mathcal{B})$, where $\mathcal{G} = \{G_1, G_2, \cdots, G_m\}$ and $|G_i| = g_i$ $(1 \leq i \leq m)$. For $i < m$, there are g_i frame parallel classes missing the group G_i, and the same number of parallel classes in the $IKPD_2(g_i + w, w)$ which contain a block of size four; match these arbitrarily, placing the g_i points of the $IKPD_2(g_i + w, w)$ on the i-th group of the frame and the w points in its hole on w new points.

Next, each $IKPD_2(g_i + w, w)$ contains $w - 3$ parallel classes of triple. From union of this with $w - 3$ holey parallel classes of the $KPD_2(g_m + w)$, to form $w - 3$ additional parallel classes. There remain g_m parallel classes of the $KPD_2(g_m + w)$, which can be matched arbitrarily with the g_m frame parallel classess of the m-th group to complete the construction.

It is easy to check that this construction gives a Kirkman Packing Design with $\sum_{1\leq i\leq m} g_i + w - 3$ resolution classes. The proof is completed.

Lemma 3.3 *If* $t \geq 5$ *and* $t \notin \{6, 10, 14, 18, 22\}$, *then there is a* $KPD_2(12t + 3k + 4)$ *for* $4 \leq k \leq t$.

Proof. We start with the resolvable $TD(5,t)$ (which existence see [4]) and give the $t - k$ points in one group weight 0 and the remaining points weight 1 to obtain a $\{4, 5, k, t\}$-GDD of type $5^k 4^{t-k}$. And then give the points of the GDD weight 3 to obtain a two-fold Kirkman Frame of type $15^k 12^{t-k}$ by Theorem 2.2. The result then follows from Lemma 3.2, the input designs $IKPD_2(16,4)$ and $IKPD_2(19,4)$ come from Example 2.3.

Lemma 3.4 *There exists a* $KPD_2(v)$ *for every* $v \in \{6t + 4 : 8 \leq t \leq 15\} \cup \{142\}$.

Proof. For $v = 52$, we start with the $TD(4,4)$ and give the points weight 3 to obtain a Kirkman Frame of type 12^4 by Theorem 2.2. The result then follows from Lemma 3.2, the input design $KPD(16)$ comes from Lemma 3.1.

For $v = 58$, we start with the $TD(5,4)$ and give the 2 points in one group weight 0 and the remaining points weight 1 to obtain a $\{4,5\}$-GDD of type $4^4 2^1$. And then give the points of the GDD weight 3 to obtain a Kirkman Frame of type $12^4 6^1$ by Theorem 2.2. The result then follows from Lemma 3.2, the input design $KPD(10)$ comes from Lemma 3.1.

For $v = 64$, 70 and 76, we start with the $TD(5,5)$ and give the s points in one group weight 0 and the remaining points weight 1, $s = 5$, 3 and 1, to obtain a $\{4,5\}$-GDD of type $5^4(5-s)^1$. And then give the points of the GDD weight 3 to obtain a Kirkman Frame of type $15^4(15-3s)^1$ by Theorem 2.2. The result then follows from Lemma 3.2, the input design $KPD(19)$ comes from Lemma 3.1.

For $v = 82$, 88 and 94, we start with the $TD(6,5)$ and give the s points in one group weight 0 and the remaining points weight 1, $s = 4$, 2 and 0, to obtain a $\{5,6\}$-GDD of type $5^5(5-s)^1$. And then give the points of the GDD weight 3 to obtain a Kirkman Frame of type $15^5(15-3s)^1$ by Theorem 2.2. The result then follows from Lemma 3.2, the input designs $KPD(7)$ and $KPD(13)$ come from Lemma 3.1.

For $v = 142$, we start with the $TD(8,7)$ and give the 3 point in one group and 1 point in each of the other groups weight 0 and the remaining points weight 1 to obtain a $\{6,7,8\}$-GDD of type $6^7 4^1$. And then give the points of the GDD weight 3 to obtain a Kirkman Frame of type $18^7 12^1$ by Theorem 2.2. The result then follows from Lemma 3.2, the input design $IKPD(22,4)$ comes from Example 2.3.

The Proof of Theorem 1.4: From Lemma 3.3 we know that the result is true for $v \geq 100$ and $v \neq 142$ or 145. For the cases $v < 100$ and $v = 142$ and 145, we know that the result is true from Lemmas 3.1 and 3.4.

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Appendix

Point Set: $Z_{v-3} \cup \{x_1, x_2, x_3\}$
Parallel Classes: Develop the following class mod $(v-3)$:

$$
\begin{array}{ll}
v = 13: & \{0,1,3,5\}, \{x_1,2,6\}, \{x_2,4,7\}, \{x_3,8,9\}. \\
v = 28: & \{0,1,9,20\}, \{2,12,23\}, \{4,16,19\}, \\
 & \{5,11,13\}, \{6,15,18\}, \{7,22,24\}, \\
 & \{x_1,3,21\}, \{x_2,7,8\}, \{x_3,10,14\}. \\
v = 34: & \{0,1,15,28\}, \{2,17,29\}, \{3,12,26\}, \{4,5,7\}, \\
 & \{6,8,30\}, \{9,19,27\}, \{13,18,24\}, \{14,20,25\}, \\
 & \{x_1,10,22\}, \{x_2,11,21\}, \{x_3,16,23\}. \\
v = 40: & \{0,19,29,36\}, \{1,4,7\}, \{2,6,11\}, \{8,10,21\}, \\
 & \{12,30,35\}, \{13,23,25\}, \{15,28,32\}, \\
 & \{16,30,31\}, \{17,22,33\}, \{18,27,34\}, \\
 & \{x_1,3,9\}, \{x_2,5,24\}, \{x_3,14,26\}. \\
v = 46: & \{0,1,3,7\}, \{2,10,15\}, \{4,13,23\}, \{5,16,28\}, \\
 & \{6,20,35\}, \{8,24,41\}, \{9,27,29\}, \{11,32,33\}, \\
 & \{12,36,39\}, \{18,22,31\}, \{19,30,37\}, \{21,26,38\}, \\
 & \{x_1,14,42\}, \{x_2,17,25\}, \{x_3,34,40\}.
\end{array}
$$