Image segmentation through operators based on topology

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Abstract. *We consider a cross-section topology that is defined on grayscale images. The main interest of this topology is that it keeps track of the grayscale information of an image. We define some basic notions relative to that topology. Furthermore, we indicate how to acquire a homotopic kernel and a leveling kernel. Such kernels can be seen as "ultimate" topological simplifications of an image. A kernel of a real image, though simplified, is still an intricated image from a topological point of view. We introduce the notion of an irregular region. The iterative removal of irregular regions in a kernel enables us to selectively simplify the topology of the image. Through an example, we show that this notion leads to a method for segmenting some grayscale images without the need to define and tune parameters.* © *1997 SPIE and IS&T.* [S1017-9909(97)00104-9]

1 Introduction

The topology of discrete binary 2-D images has received a lot of attention (see Ref. 1). In Refs. 2 and 3, a fuzzy digital topology for grayscale images is considered. Nevertheless, as far as we know, no systematic study of the homotopic transformations of 2-D grayscale images has been made.

In this paper, we introduce some basic topological notions for 2-D grayscale images. For that purpose, we define a *cross-section* topology. The homotopic transformations relative to that topology preserve the main grayscale information of an image. This topology was considered in Ref. 4, but it has not been developed. We give the basic definitions that enables us to use this topology. We introduce some *topological numbers* that lead to a classification of points according to their topological characteristics. We give a necessary and sufficient local condition for changing the value of a point without altering the topology of an image. From these notions, we define the *homotopic kernel* and the *leveling kernel* of an image. These kernels can be seen as "ultimate" grayscale thinnings[5] of an image.

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We use these basic topological notions to develop a new method for segmenting an image into regions. A major feature of this method is that it does not involve any parameter. Let us consider a grayscale image as a relief.[6] Suppose the image is a "two-class" image, i.e., it consists in objects that lie on a background. The objects we want to extract can be seen as "significant basins" in this relief.[4,7–12] The homotopic kernel of such an image, though very simplified, is still an intricated image. Since the topology has been preserved, all basins (significant and nonsignificant) remain. We define two *regularization operators* that alter the topology of the image by deleting two kinds of nonsignificant regions. After the application of these operators, we can extract a binary image that has the main topological features of the desired result. Finally, we present some *reconstruction operators* that perform a "conditional thickening" of the preceding binary image to obtain the final segmented image.

The paper is organized as follows: Section 2 introduces some notions concerning relations, which are used as a general framework for presenting operators. Section 3 reviews the definition of topology in binary images. In Sections 4 and 5, we define some basic notions for grayscale images, and introduce the cross-section topology. In Sections 6, 7 and 8, we present the basic transformations based on topology that we use to segment some grayscale images. The effect of these transformations on several "real world" images is also presented.

2 Relations

In this section, we give some basic notations concerning relations. Relations are used in a classical way to define some topological notions, such as the notions of path, plateau, and regional minimum. Furthermore, all the operators presented in this paper are defined in terms of relations: a "lower" ("upper") operator consists in decreasing (increasing) the value of a point that satisfies a given condition, and this process is repeated until stability is achieved. The result obtained can be viewed as a "kernel" of a relation.

Let E be a set, $\mathscr{E}$ be the set composed of all subsets of E, and let $\mathscr{R}$ be a (binary) relation on E, i.e., $\mathscr{R}$ is a set of ordered pairs of elements of E. We know that a relation $\mathscr{R}$ can also be defined as a mapping from E to $\mathscr{E}$; the correspondence between this mapping, also denoted $\mathscr{R}$, and the ordered pairs is given by $\forall u \in E$, $\forall v \in E$, $v \in \mathscr{R}(u) \Leftrightarrow (u,v) \in \mathscr{R}$. In this paper, relations are considered as mappings. A sequence $u_0,\ldots,u_k$ such that $u_i \in \mathscr{R}(u_{i-1})$ is an *$\mathscr{R}$-path from u_0 to u_k*; the *length* of this path is k. We denote $\mathscr{R}^k$, the relation defined by:

$v \in \mathscr{R}^k(u) \Leftrightarrow$ there is an $\mathscr{R}$-path of length k from u to v.

We denote $\mathscr{R}^\infty$ the transitive closure of $\mathscr{R}$, i.e., the relation defined by:

$v \in \mathscr{R}^\infty(u) \Leftrightarrow$ there is an $\mathscr{R}$-path from u to v.

We say that v is *a kernel of u for $\mathscr{R}$* if $v \in \mathscr{R}^\infty(u)$ and $\mathscr{R}(v)=\emptyset$. We say that v is *a kernel for $\mathscr{R}$* if $\mathscr{R}(v)=\emptyset$. If $\mathscr{R}_1$ and $\mathscr{R}_2$ are relations, the relation $\mathscr{R}_1 \cup \mathscr{R}_2$ is defined by $v \in [\mathscr{R}_1 \cup \mathscr{R}_2](u) \Leftrightarrow v \in \mathscr{R}_1(u)$ or $v \in \mathscr{R}_2(u)$. Let $U \subset E$. The set $\mathscr{R}(U)$ is defined by $\mathscr{R}(U) = \cup\{\mathscr{R}(u), u \in U\}$.

3 Topology of Binary Images

In this section we review the basic notions of topology for binary images (see Ref. 1).

We denote $\mathbb{Z}$ as the set of relative integers. A point $x \in \mathbb{Z}^2$ is defined by (x_1,x_2) with $x_i \in \mathbb{Z}$. We consider the two relations Γ_4 and Γ_8, which are relations on $\mathbb{Z}^2$ and which define two neighborhoods of a point $x \in \mathbb{Z}^2$:

$$\Gamma_4(x)=\{y \in \mathbb{Z}^2; |y_1-x_1|+|y_2-x_2| \leq 1\},$$

$$\Gamma_8(x)=\{y \in \mathbb{Z}^2; \max(|y_1-x_1|,|y_2-x_2|) \leq 1\}.$$

In the following, we will denote n the number such that $n=4$ or $n=8$. We define $\Gamma_n^*(x)=\Gamma_n(x)\backslash\{x\}$. The point $y \in \mathbb{Z}^2$ is *n-adjacent* to $x \in \mathbb{Z}^2$ if $y \in \Gamma_n^*(x)$. An *n-path* is a path for the relation $\mathscr{R}=\Gamma_n^*$.

Let $X \subset \mathbb{Z}^2$ and $x \in X$; we define $X_n(x)=\Gamma_n(x)\cap X$. The relation $\mathscr{R}=[X_n]^\infty$ is an equivalence relation. The *n-connected components of X* (or the *n-components of X*) are the equivalence classes of this relation. The set composed of all n-connected components of X n-adjacent to a point x is denoted $C_n[x,X]$. Note that $C_n[x,X]$ is a set of subsets of $\mathbb{Z}^2$, not a set of points. To have a correspondence between the topology of X and that of $\bar{X}$, we must consider two different kinds of adjacencies for X and $\bar{X}$: if we use the n-adjacency for X, we must use the $\bar{n}$-adjacency for $\bar{X}$, with $(n,\bar{n})=(8,4)$ or $(4,8)$. If $X \subset \mathbb{Z}^2$ is finite, the infinite connected component of $\bar{X}$ is the background, the other components are the holes of X.

Let $X \subset \mathbb{Z}^2$ and $x \in \mathbb{Z}^2$, the two topological numbers are (#X stands for the cardinal of X):

$$T(x,X)=\#C_n[x,\Gamma_8^*(x)\cap X]; \bar{T}(x,X)=\#C_{\bar{n}}[x,\Gamma_8^*(x)\cap \bar{X}].$$

We say that $x \in X$ is an *isolated point* if $T(x,X)=0$, a *border point* if $\bar{T}(x,X)>0$, and an *interior point* if $\bar{T}(x,X)=0$. The point $x \in X$ is *simple (for X)* if there is a one to one correspondence between the n-components of X and those of $X\backslash\{x\}$ and also between the $\bar{n}$-components of $\bar{X}$ and those of $\bar{X}\cup\{x\}$. The point $x \in \bar{X}$ is *simple (for X)* if there is a one to one correspondence between the n-components of X and those of $X\cup\{x\}$ and also between the $\bar{n}$-components of $\bar{X}$ and those of $\bar{X}\backslash\{x\}$. The set Y is *lower homotopic to X* if Y can be obtained from X by iterative deletion of simple points. The set Y is *upper homotopic to X* if Y can be obtained from X by iterative addition of simple points. Two sets X and Y are *homotopic* if Y can be obtained from X by iterative deletions or additions of simple points.

The following property is fundamental for our purpose, since it enables us to locally characterize simple points. We propose a similar property for grayscale images.

$x \in \mathbb{Z}^2$ is simple $\Leftrightarrow T(x,X)=1$ and $\bar{T}(x,X)=1$.

4 Basic Notions for Grayscale Images

A 2-D grayscale image may be seen as an application F from $\mathbb{Z}^2$ to $\mathbb{Z}$. For each point $x \in \mathbb{Z}^2$, $F(x)$ is the gray-level value of x. We denote $\mathscr{F}$ as the set composed of all applications from $\mathbb{Z}^2$ to $\mathbb{Z}$.

Definition 1. Let $F \in \mathscr{F}$. We define the relation $F_n^=$ by: $\forall x \in \mathbb{Z}^2$, $\forall y \in \mathbb{Z}^2$, $y \in F_n^=(x) \Leftrightarrow y \in \Gamma_n(x)$ and $F(x)=F(y)$. An $\mathscr{R}$-path with $\mathscr{R}=F_n^=$ is called a *constant n-path*. A set $X \subset \mathbb{Z}^2$ is an *n-plateau (for F)* if X is an equivalence class for the equivalence relation $\mathscr{R}=[F_n^=]^\infty$.

As in the binary case, we use two different kinds of adjacencies (n and $\bar{n}$) for grayscale images. An n-adjacency is used for regional maxima, while an $\bar{n}$-adjacency is used for regional minima, with $(n,\bar{n})=(4,8)$ or $(8,4)$; a regional maximum (regional minimum) being a set of points of uniform altitude with only lower (higher) neighbors.

Definition 2. Let $F \in \mathscr{F}$. We define the following relations on $\mathbb{Z}^2$, $\forall x \in \mathbb{Z}^2$, $\forall y \in \mathbb{Z}^2$:

$y \in F^{++}(x) \Leftrightarrow y \in \Gamma_n(x)$ and $F(x)<F(y)$;

$y \in F^{+}(x) \Leftrightarrow y \in \Gamma_n(x)$ and $F(x)\leq F(y)$;

$y \in F^{--}(x) \Leftrightarrow y \in \Gamma_{\bar{n}}(x)$ and $F(x)>F(y)$;

$y \in F^{-}(x) \Leftrightarrow y \in \Gamma_{\bar{n}}(x)$ and $F(x)\geq F(y)$.

(a)	(b)	(c)	(d)	(e)	(f)	(g)
30 30 30	30 30 30	30 30 30	30 30 10	10 10 60	10 10 60	40 40 40
30 30 20	30 30 10	30 40 10	30 10 40	10 40 30	10 40 30	40 30 10
30 10 10	30 10 30	30 10 30	30 40 30	40 40 30	50 50 30	40 10 10

Fig. 1 Examples of configurations of the neighborhood of a point: (a) maximal destructible point, (b) maximal divergent point, (c) peak, (d) well, (e) constructible divergent point, (f) saddle point, and (g) simple side.

An $\mathscr{R}$-path with $\mathscr{R}=F^{++}$ (Resp. $\mathscr{R}=F^{+}$, $\mathscr{R}=F^{--}$, $\mathscr{R}=F^{-}$) is called a *strictly increasing* (resp. *increasing*, *strictly decreasing*, *decreasing*) path. A *(regional) maximum* [*(regional) minimum*] of F is a set $X\subset\mathscr{Z}^2$ such that X is an n-plateau ($\bar{n}$-plateau) for F with $F^+(X)=\cup\{F^+(x),x\in X\}=X$ $[F^-(X)=\cup\{F^-(x),x\in X\}=X]$. An $\bar{n}$-plateau (n-plateau) $X\subset\mathscr{Z}^2$ is a *lower region* (*upper region*) of F if it is not a maximum (minimum) of F. The *upper set* (*lower set*) of F is the set composed of all points belonging to upper regions (lower regions) of F.

Our goal is to define some transformations changing a mapping $F\in\mathscr{F}$, which represents a grayscale image, into another mapping $G\in\mathscr{F}$. We introduce the notation of a "pointwise transformation" as a simple such transformation, from which we build more complex ones. The result G of this transformation is obtained by replacing the value $F(x)$ of a given point x by a certain value v, the value of all other points being unchanged.

Definition 3. Let $F\in\mathscr{F}$, $x\in\mathscr{Z}^2$ and $v\in\mathscr{Z}$. We denote $[F;F(x):=v]$ the element G of $\mathscr{F}$ such that $G(x)=v$ and $\forall y\neq x$, $G(y)=F(y)$.

5 Cross-Section Topology

We now introduce the cross-section topology. Let $F\in\mathscr{F}$. The *section* of F at the level k is the set composed of all points x such that $F(x)\geqslant k$. Observe that a section is a binary set. A transformation on F will be "topology preserving" if the topology of all the sections of F is preserved. Thus, the cross-section topology of mappings may be directly derived from the topology of binary sets.

Definition 4. Let $F\in\mathscr{F}$, we denote $F_k=\{x\in\mathscr{Z}^2;F(x)\geqslant k\}$ with $k\in\mathscr{Z}$; F_k is called a *section of* F. The point $x\in\mathscr{Z}^2$ is *destructible* (for F) if x is simple for F_k, with $k=F(x)$. The point $x\in\mathscr{Z}^2$ is *constructible* (for F) if x is simple for F_{k+1}, with $k=F(x)$. We define the two relations *THIN* and *THICK* defined on $\mathscr{F}$; $\forall F\in\mathscr{F}$ and $\forall G\in\mathscr{F}$, $G\in THIN(F)\Leftrightarrow\exists x\in\mathscr{Z}^2$, such that x is destructible for F and $G=[F;F(x):=F(x)-1]$; $G\in THICK(F)\Leftrightarrow\exists x\in\mathscr{Z}^2$, such that x is constructible for F and $G=[F;F(x):=F(x)+1]$.

Note that if $F'\in THIN(F)$ or $F'\in THICK(F)$, then every section F'_k of F' is homotopic, in the binary sense, to the corresponding section F_k of F. This leads to the notion of homotopy on $\mathscr{F}$.

Definition 5. Let $F\in\mathscr{F}$ and $G\in\mathscr{F}$. Here G is *lower homotopic* to F if $G\in THIN^\infty(F)$, G is *upper homotopic* to F if $G\in THICK^\infty(F)$, and F and G are *homotopic* if $G\in[THIN\cup THICK]^\infty(F)$.

The following definition introduces the fundamental neighborhoods that must be used to handle the topology of grayscale images, as well as topological numbers that describe the topological characteristics of a point (we follow an approach used in the 3-D case, see Ref. 13).

Definition 6. Let $F\in\mathscr{F}$ and $x\in\mathscr{Z}^2$. We define the four neighborhoods:

$$\Gamma^{++}(x,F)=\{y\in\Gamma_8^*(x),F(y)>F(x)\};$$

$$\Gamma^{+}(x,F)=\{y\in\Gamma_8^*(x),F(y)\geqslant F(x)\};$$

$$\Gamma^{--}(x,F)=\{y\in\Gamma_8^*(x),F(y)<F(x)\};$$

$$\Gamma^{-}(x,F)=\{y\in\Gamma_8^*(x),F(y)\leqslant F(x)\}.$$

We also define the four *topological numbers*:

$$T^{++}(x,F)=\#C_n[x,\Gamma^{++}(x,F)];$$

$$T^{+}(x,F)=\#C_n[x,\Gamma^{+}(x,F)];$$

$$T^{--}(x,F)=\#C_{\bar{n}}[x,\Gamma^{--}(x,F)];$$

$$T^{-}(x,F)=\#C_{\bar{n}}[x,\Gamma^{-}(x,F)].$$

When there is no confusion, we denote $T^{++}=T^{++}(x,F)$, $T^{+}=T^{+}(x,F)$, $T^{-}=T^{-}(x,F)$, and $T^{--}=T^{--}(x,F)$.

The following property can be directly derived from the preceding definitions and the characterization of simple points in binary sets. It shows that the topological numbers enable to locally characterize constructible and destructible points.

Property 1. Let $F\in\mathscr{F}$ and $x\in\mathscr{Z}^2$.

x is destructible for $F\Leftrightarrow T^{+}=1$ and $T^{--}=1$;

x is constructible for $F\Leftrightarrow T^{++}=1$ and $T^{-}=1$.

Furthermore, the topological numbers enable a classification of the topological characteristics of a point (see the examples in Fig. 1, with $n=8$).

20	20	20	20	20	20	20	20	20	20	20	20	20
20	25	25	45	50	55	60	50	45	45	25	25	20
20	25	50	20	20	20	20	35	20	20	45	25	20
20	45	20	20	35	20	20	60	55	20	20	45	20
20	45	20	35	20	35	20	45	50	20	20	50	20
20	45	20	20	35	20	20	45	20	20	20	45	20
20	60	20	20	20	20	20	20	35	50	35	45	20
20	45	20	20	20	70	20	20	20	20	20	50	20
20	45	20	20	60	50	60	20	25	30	30	45	20
20	25	45	20	20	60	20	20	35	35	45	25	20
20	25	25	45	45	50	50	60	50	45	25	25	20
20	20	20	20	20	20	20	20	20	20	20	20	20

(a)

20	20	20	20	20	20	20	20	20	20	20	20	20
20	50	50	45	60	60	60	60	60	60	45	50	20
20	50	50	20	20	20	35	35	35	35	45	50	20
20	50	20	20	35	20	20	60	60	35	50	50	20
20	45	20	35	20	35	20	60	60	20	50	50	20
20	60	20	20	35	35	20	60	35	35	35	45	20
20	60	60	20	20	20	20	35	35	50	35	50	20
20	60	60	45	70	70	70	50	35	35	35	50	20
20	60	45	45	70	50	70	50	60	45	50	50	20
20	60	45	70	70	60	50	50	60	45	50	50	20
20	60	45	70	70	70	50	60	60	45	50	50	20
20	20	20	20	20	20	20	20	20	20	20	20	20

(b)

20	20	20	20	20	20	20	20	20	20	20	20	20
20	20	20	45	45	45	60	45	45	45	20	20	20
20	20	50	20	20	20	20	35	20	20	45	20	20
20	45	20	20	35	20	20	60	20	20	20	45	20
20	45	20	35	20	35	20	35	20	20	20	50	20
20	45	20	20	35	20	20	35	20	20	20	45	20
20	60	20	20	20	20	20	20	35	50	35	45	20
20	45	20	20	20	70	20	20	20	20	20	50	20
20	45	20	20	60	50	60	20	20	20	20	45	20
20	20	45	20	20	60	20	20	20	20	45	20	20
20	20	20	45	45	20	50	60	45	45	20	20	20
20	20	20	20	20	20	20	20	20	20	20	20	20

(c)

Fig. 2 Homotopic transformations: (a) original image; (b) upper kernel; and (c) lower kernel.

Definition 7. Let $F \in \mathscr{F}$ and $x \in \mathbb{Z}^2$: x is a peak if $T^+ = 0$; x is *minimal* if $T^{--} = 0$; x is *divergent* if $T^{--} > 1$; x is a well if $T^- = 0$; x is maximal if $T^{++} = 0$; x is *convergent* if $T^{++} > 1$; x is a *lower point* if it is not maximal; x is an *upper point* if it is not minimal; x is an *interior point* if it is minimal and maximal; x is a *simple side* if it is destructible and constructible; x is a *saddle point* if it is divergent and convergent.

By considering all the possible values of the four topological numbers we can see the following.

Property 2. Let $F \in \mathscr{F}$ and $x \in \mathbb{Z}^2$, x corresponds necessarily to one and only one of the following types: (1) a peak, (2) a well, (3) an interior point, (4) a minimal constructible point, (5) a maximal destructible point, (6) a minimal convergent point, (7) a maximal divergent point, (8) a simple side, (9) a destructible convergent point, (10) a constructible divergent point, and (11) a saddle point.

6 Homotopic Transformations and Leveling Transformations

We introduce the following two basic transformations relative to the cross-section topology.

Definition 8. Let $F \in \mathscr{F}$. A kernel of F for the relation *THIN* is called a *lower homotopic kernel* of F. A kernel of F for the relation *THICK* is called an *upper homotopic kernel* of F.

The lower (upper) homotopic kernel of F can be seen as an "ultimate" topological simplification of F, in the sense that no destructible (constructible) point remains in the kernel. We can compute a lower homotopic kernel of F by using iteratively the definition of $THIN(F)$ (Definition 4). At each step of the procedure, we lower the value of a destructible point by 1. In fact, it is possible to have a faster procedure based on the following properties.

Let $F \in \mathscr{F}$ and $x \in \mathbb{Z}^2$. We define:

$$\alpha^{++}(x,F) = \min\{F(y), y \in \Gamma^{++}(x,F)\} \text{ if } \Gamma^{++}(x,F) \neq \emptyset, \alpha^{++}(x,F) = F(x) \text{ otherwise;}$$

$$\alpha^{--}(x,F) = \max\{F(y), y \in \Gamma^{--}(x,F)\} \text{ if } \Gamma^{--}(x,F) \neq \emptyset, \alpha^{--}(x,F) = F(x) \text{ otherwise.}$$

If x is constructible for F, then F and $[F;F(x):=\alpha^{++}(x,F)]$ are homotopic. If x is destructible for F, then F and $[F;F(x):=\alpha^{--}(x,F)]$ are homotopic.

Lower and upper kernels can be efficiently computed using a "breadth-first" strategy. A classical technique for implementing such a strategy is based on two lists of points. The first list is initialized with all constructible (destructible) points of the original image. If the first list is not empty, we extract the first point of the list. If this point is constructible (destructible), its value is changed in the image, and the neighbors of this point are inserted in the second list. After scanning the first list, we exchange the roles of the first and the second lists. We repeat the procedure until the second list is empty. All kernels presented in this paper were computed using this technique.

In Fig. 2, an upper and a lower homotopic kernel are represented (with $n=8$). Two kernels corresponding to a real image are also represented in Fig. 3. Figure 3(a) shows an original image (256 graylevels); the lower levels are in black and the higher levels are in white. Below each grayscale image of Fig. 3 (and later in Figs. 6, 8, and 10), the corresponding regional minima are given; they appear in white. We can see that, for a real image, there are a lot of minima, each of these regions being composed of only few points. Because of their size, only a few points of the minima have been "deleted" by the upper kernel transformation [Fig. 3(b)]. On the other hand, the lower kernel transformation expands the minima as much as possible. Nevertheless, if we examine the upper set (nonminimal plateaus) of Fig. 3(c), we notice that it is not thin. In fact, an homotopic lower kernel cannot be viewed as a "network of

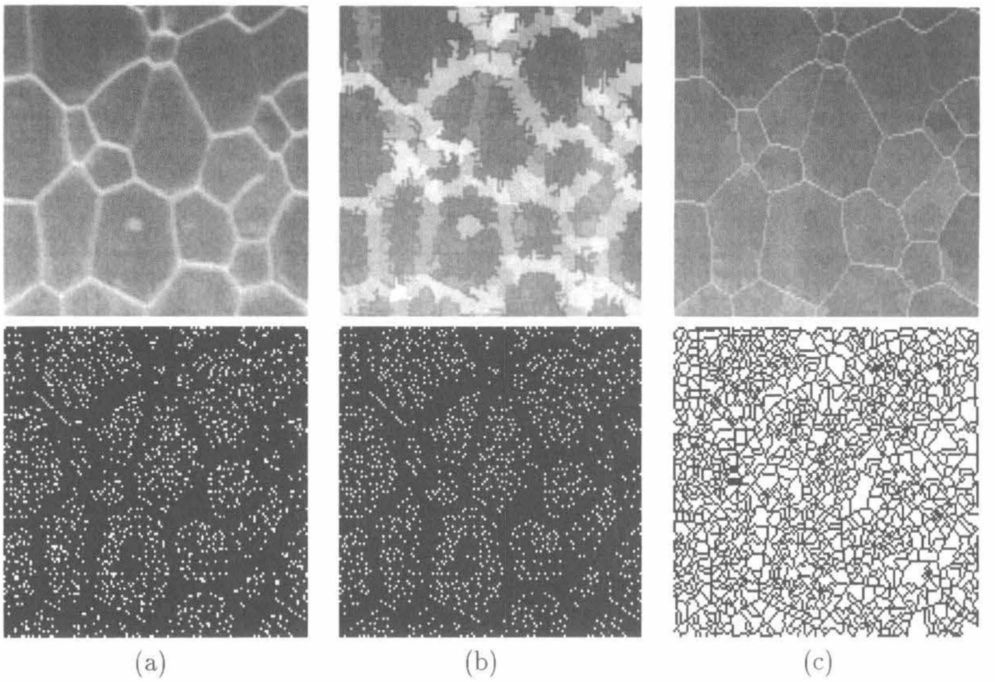

Fig. 3 Homotopic transformations: (a) original image; (b) upper kernel; and (c) lower kernel.

thin lines'': due to the discrete nature of the image representation, it may contain thick parts. Four basic configurations, depicted Fig. 4, may explain this phenomenon:

1. The *flat junction*; we retrieve this configuration in homotopic kernels of binary images.
2. The *grayscale junction*; this configuration is thicker than a binary kernel, in the sense that there are some points that are simple for the (binary) upper set.
3. The *network of minima*; the points adjacent to some very close regional minima (often one point minima)

30	30	30	30	30	30	30	30	30	30	30	30	30
30	10	10	30	10	10	30	10	10	30	10	10	30
30	10	10	10	30	10	30	10	30	10	10	10	30
30	10	10	10	10	30	30	30	10	10	10	10	30
30	30	30	30	30	30	30	30	30	30	30	30	30
30	10	10	10	10	30	30	30	10	10	10	10	30
30	10	10	10	30	10	30	10	30	10	10	10	30
30	10	10	30	10	10	30	10	10	30	10	10	30
30	30	30	30	30	30	30	30	30	30	30	30	30

(a)

40	40	40	40	40	40	40	40	40	40	40	40	40
40	10	10	10	10	10	10	10	10	10	10	10	40
40	10	10	10	10	10	10	10	10	10	10	10	40
40	10	10	10	10	10	40	10	10	10	10	10	40
40	40	40	40	40	40	30	40	40	40	40	40	40
40	10	10	10	10	10	30	10	10	10	10	10	40
40	10	10	10	10	10	30	10	10	10	10	10	40
40	10	10	10	10	10	30	10	10	10	10	10	40
40	40	40	40	40	40	40	40	40	40	40	40	40

(b)

40	40	40	40	40	40
40	30	30	30	10	40
40	30	30	10	30	40
40	30	10	30	30	40
40	10	30	30	30	40
40	40	40	40	40	40

(c)

30	30	30	30	30	30	30	30	30	30	30	30	30
30	10	10	10	10	10	10	10	10	10	10	10	30
30	10	10	10	10	10	10	10	10	10	10	10	30
30	10	10	10	10	40	40	40	10	10	10	10	30
30	10	10	10	40	30	30	30	40	10	10	10	30
30	10	10	10	40	30	30	30	40	10	10	10	30
30	10	10	10	40	30	30	30	40	10	10	10	30
30	10	10	40	20	40	30	40	20	40	10	10	30
30	10	10	10	40	10	30	10	40	10	10	10	30
30	10	10	10	10	10	30	10	10	10	10	10	30
30	10	10	10	10	10	30	10	10	10	10	10	30
30	30	30	30	30	30	30	30	30	30	30	30	30

(d)

Fig. 4 Four basic configurations of thick lower kernels: (a) flat junction, (b) grayscale junction, (c) network of minima, and (d) fortified castle.

20	20	20	20	20	20	20	20	20	20	20	20	20
20	20	20	45	45	45	45	45	45	45	20	20	20
20	20	45	20	20	20	20	35	20	20	45	20	20
20	45	20	20	35	20	20	35	20	20	20	45	20
20	45	20	35	20	35	20	35	20	20	20	45	20
20	45	20	20	35	20	20	35	20	20	20	45	20
20	45	20	20	20	20	20	20	35	35	35	45	20
20	45	20	20	20	60	20	20	20	20	20	45	20
20	45	20	20	60	50	60	20	20	20	20	45	20
20	20	45	20	20	60	20	20	20	20	45	20	20
20	20	20	45	45	20	45	45	45	45	20	20	20
20	20	20	20	20	20	20	20	20	20	20	20	20

(a)

20	20	20	20	20	20	20	20	20	20	20	20	20
20	20	20	45	45	45	45	45	45	45	20	20	20
20	20	45	20	20	20	20	20	20	20	45	20	20
20	45	20	20	20	20	20	20	20	20	20	45	20
20	45	20	20	20	20	20	20	20	20	20	45	20
20	45	20	20	20	20	20	20	20	20	20	45	20
20	45	20	20	20	20	20	20	20	20	20	45	20
20	45	20	20	20	60	20	20	20	20	20	45	20
20	45	20	20	60	50	60	20	20	20	20	45	20
20	20	45	20	20	60	20	20	20	20	45	20	20
20	20	20	45	45	20	45	45	45	45	20	20	20
20	20	20	20	20	20	20	20	20	20	20	20	20

(b)

20	20	20	20	20	20	20	20	20	20	20	20	20
20	20	20	45	45	45	45	45	45	45	20	20	20
20	20	45	20	20	20	20	20	20	20	45	20	20
20	45	20	20	20	20	20	20	20	20	20	45	20
20	45	20	20	20	20	20	20	20	20	20	45	20
20	45	20	20	20	20	20	20	20	20	20	45	20
20	45	20	20	20	20	20	20	20	20	20	45	20
20	45	20	20	20	20	20	20	20	20	20	45	20
20	45	20	20	20	20	20	20	20	20	20	45	20
20	20	45	20	20	45	20	20	20	20	45	20	20
20	20	20	45	45	20	45	45	45	45	20	20	20
20	20	20	20	20	20	20	20	20	20	20	20	20

(c)

Fig. 5 Leveling and regularization: (a) lower leveling; (b) lower regularization; and (c) upper regularization.

may be nondestructible. Thus, we can generate a thick plateau that is not minimal [the 30s of Fig. 4(c)]. It can be seen that networks of minima may generate thick regions of arbitrary size.

4. The *fortified castle*; the entrance of the castle consists in a thin line which comes out onto the inside of the castle [the 3×3 block of 30s in Fig. 4(d)]. The inside of a castle is not a subset of a minimum and it can be seen that we can build castles the insides of which have an arbitrary size.

We introduce now the leveling kernels that can be viewed as filtered homotopic kernels:

Definition 9. We define the *lower leveling relation* (*LLE*) and the *upper leveling relation* (*ULE*); $\forall F \in \mathcal{F}$, $\forall G \in \mathcal{F}$: $G \in LLE(F) \Leftrightarrow \exists x \in \mathbb{Z}^2$, such that x is destructible or a peak for F, and $G=[F;F(x):=F(x)-1]$; $G \in ULE(F) \Leftrightarrow \exists x \in \mathbb{Z}^2$, such that x is constructible or a well for F, and $G=[F;F(x):=F(x)+1]$. A *lower leveling kernel* is a kernel for *LLE*. An *upper leveling kernel* is a kernel for *ULE*.

As for homotopic kernels, it is possible to increase (decrease) the value of a point up to $\alpha^{++}(x,F)$ [down to $\alpha^{--}(x,F)$] for a faster procedure. Here again, we use the already mentioned breadth-first strategy for computing lower and upper leveling kernels. Let us compare the lower homotopic kernel [Fig. 2(c)] of an original image [Fig. 2(a)] with the lower leveling kernel [Fig. 5(a)] of the same image. It can be seen that the set composed of upper regions has been flattened down. A lower leveling kernel of the real image of Fig. 3(a) is depicted Fig. 6(a). Despite the appearance, this kernel is very different from the homotopic kernel of Fig. 3(c): as mentioned, the values of the points belonging to upper regions have been smoothed. This characteristic of leveling kernels is used in the next section for regularization operators.

7 Regularization Transformations

We have seen, through an example, that the minima of a real image do not correspond to the significant basins. In Fig. 3(a), the significant basins that are perceived, correspond to the cells of the image. Nevertheless, each cell contains a lot of minima. The homotopic and leveling kernels of an image, though simplified, keep all these minima. This oversegmentation problem is also crucial when using methods based on the watershed transformation.[7,9,14,15]

In this section, we propose a method for detecting significant basins. In a lower kernel, the minima come into contact through upper points. We take advantage of this feature to characterize some nonsignificant regions. Since its upper values have been flattened, we consider the lower leveling kernel rather than the lower homotopic kernel.

Definition 10. Let $K \in \mathcal{F}$ be a lower leveling kernel. Let R_1 and R_2 be two minima of K. We say that a point x is a *separating* point for R_1 and R_2, if x is adjacent to both R_1 and R_2. If R_1 and R_2 are separated by a point, we say that R_1 and R_2 are *neighboring minima*. If R is a minimum, we define the *upper value* $\Psi(R)$ of R as:

$$\Psi(R)=\max\{K(x),\ \text{for each point } x \text{ adjacent to } R\}.$$

We now introduce the notion of regularization (see Fig. 7).

Definition 11. Let $K \in \mathcal{F}$ be a lower leveling kernel. Let R be a minimum and let x be an upper point adjacent to R. We say that x is an *irregular upper point* if $K(x)-K(R)<\Psi(R)-K(x)$. In this case, $K(R)$ is called a *regularized value of* x. We say that R is an *irregular minimum* if there is a minimum R' separated from R by a point x such that $K(x)-K(R)<K(R)-K(R')$. In this case, $K(x)$ is called a *regularized value of* R. We denote $lr(K)$ $[ur(K)]$ an element of $\mathcal{F}$ obtained from K by replacing all the values of upper irregular points (the values of irregular minima) by their regularized values. The *upper regularization relation* *UR* is defined by:

$$K' \in UR(K) \Leftrightarrow K' \text{ is a lower leveling kernel of } ur(K).$$

The *lower regularization relation LR* is defined by:

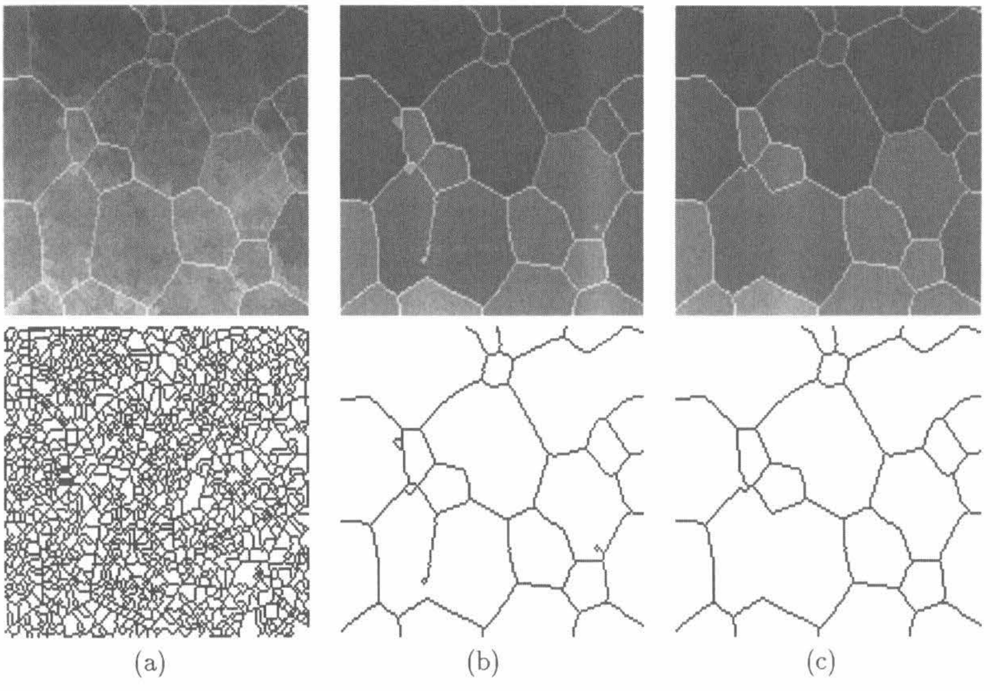

Fig. 6 Leveling transformation and regularization: (a) lower leveling; (b) lower regularization; and (c) upper regularization.

$K' \in LR(K) \Leftrightarrow K'$ is a lower leveling kernel of $lr(K)$.

An *upper regularized kernel* is a kernel for *UR*. A *lower regularized kernel* is a kernel for *LR*.

It is possible to get a lower regularized kernel of an image by regularizing all the irregular upper points of the image and by computing a lower leveling kernel of the result. We have to repeat this procedure until stability. The same approach is used for upper regularized kernels.

A lower regularized kernel of the lower leveling kernel of Fig. 5(a) is shown in Fig. 5(b). The upper regularized kernel of Fig. 5(b) is shown in Fig. 5(c). The regularization transformations of the leveling kernel of Fig. 6(a) are given in Figs. 6(b) and 6(c). In Fig. 8, the lower leveling transformation of a 256 gray-level image is depicted, as well as the result after regularization. Note the noticeable gradient of illumination of the original image. We see that almost all significant basins have been detected and that the minima of the final result (image below) can be seen as the segmentation of the original image. The result can be further improved using a hierarchical regularization technique (see Ref. 16). Note that there was no preprocessing of the original images of Figs. 3 and 8; the only transformations that were applied are (1) lower leveling transformation, (2) lower regularization, (3) upper regularization, and (4) extraction of minima.

8 Reconstruction Transformation

The image F obtained after regularization is a "thin" image. The minima of this image correspond to the segmentation of the original image I, up to a homotopic transformation. In fact, the significant basins of the original image have been made as broad as possible by the thinning process. To recover the shape of the original basins as much as possible, we require a transformation that realizes a conditional thickening of the upper set X of F (see Fig. 9).

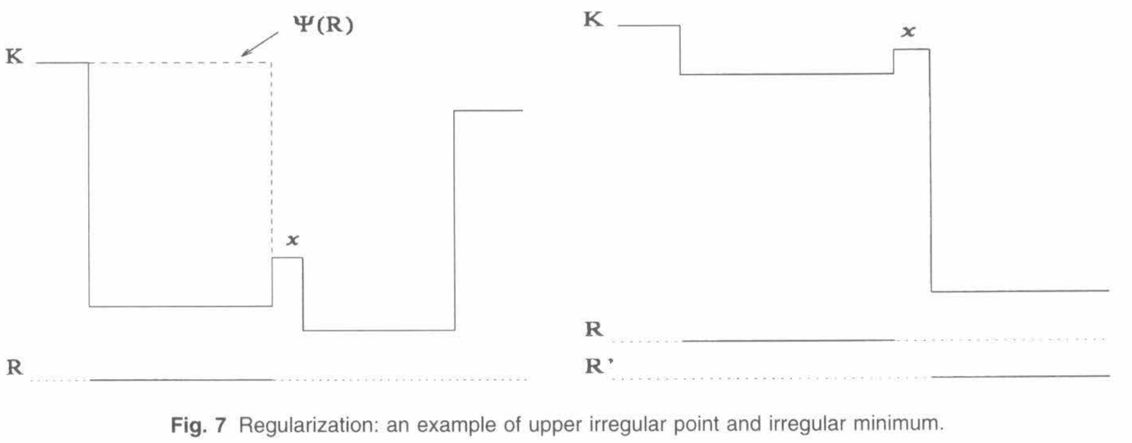

Fig. 7 Regularization: an example of upper irregular point and irregular minimum.

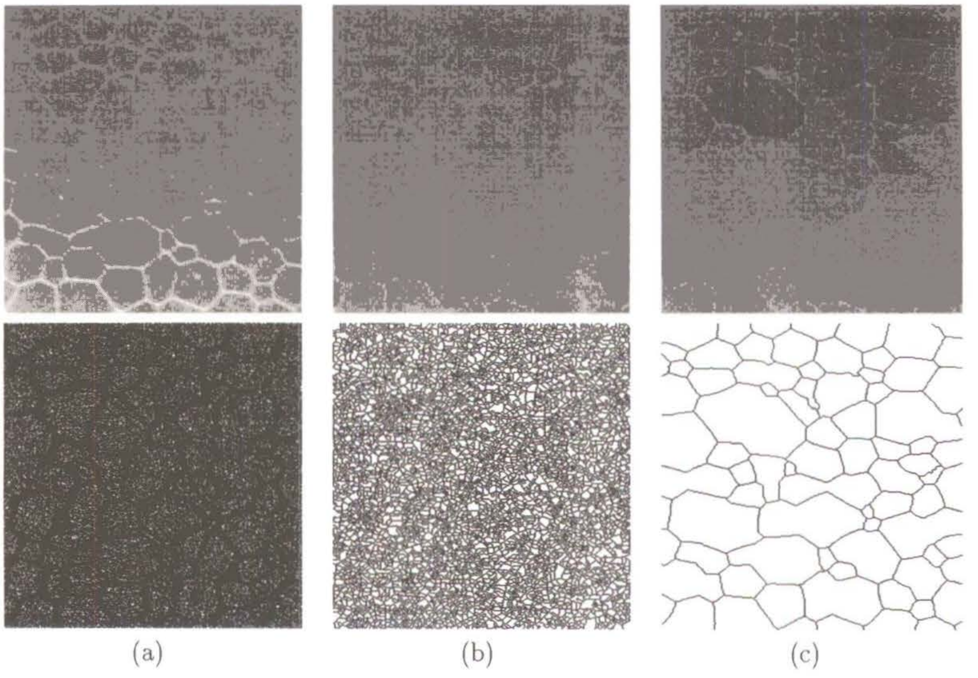

Fig. 8 Illustration of regularization: (a) original image; (b) lower leveling; and (c) regularization.

Definition 12. Let $I \in \mathcal{F}$ and $F \in \mathcal{F}$. We denote $\mathcal{X}$ as the set composed of all subsets of $\mathbb{Z}^2$. We define the relation $UR_{I,F}$, which is a relation on $\mathcal{X}$;

$\forall X \in \mathcal{X}, \forall Y \in \mathcal{X}: Y \in UR_{I,F}(X) \Leftrightarrow$

1. $\exists x \in \bar{X}$ such that x is simple for X and x belongs to a minimum R of F; and
2. $Y = X \cup \{x\}$; and
3. $|I(x) - \Psi(R)| < |I(x) - F(x)|$.

An *upper reconstruction kernel of F inside I* is a kernel of the upper set of F for the relation $UR_{I,F}$.

The original image (256 gray-levels) depicted in Fig. 10(a) contains a lot of minima (the image below). The result after lower and upper regularization performed on a lower leveling kernel is almost perfect [Fig. 10(b)]: we recover the topology of the significant basins of the original image. The binary set obtained after reconstruction [Fig. 10(c)] recovers the shape of these basins.

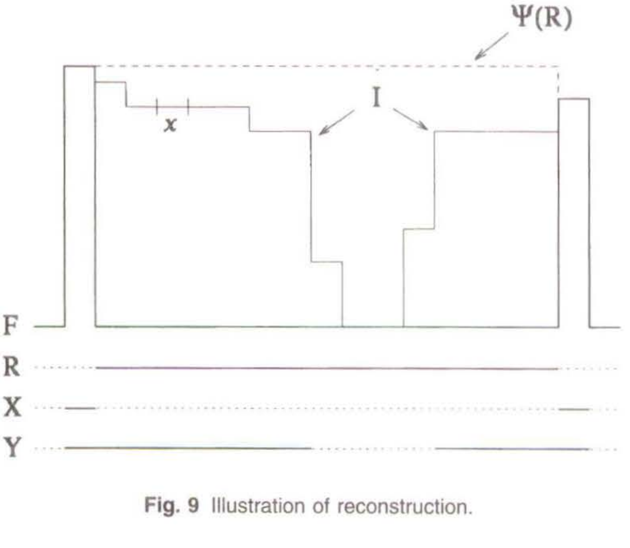

Fig. 9 Illustration of reconstruction.

In Fig. 11(a), a magnetic resonance (MR) image of a human brain is depicted. The segmentation must bring to the fore the folds of the brain. Fig. 12(a) is a picture of an electrophoresis gel. The images obtained after regularization are shown Figs. 11(b) and 12(b). Figs. 11(d) and 12(d) show the result after reconstruction. We can observe some ill-shaped lines that alter the quality of the result. They are due to the fact that a large minimum of a regularized image may contain, in the original image, several reconstructible zones. Since the reconstruction operator preserves the homotopy, some paths must be preserved between these zones. The location of these paths depends on the scanning order of the points and thus does not take into account the relief of the original image.

A solution to this problem is to consider a *hierarchical reconstruction operator* that processes the points according to their increasing gray-levels. The result of this operator is represented Figs. 11(e) and 12(e). The paths generated for the preservation of homotopy are made of points of lowest possible values. Thus the segmented image ''fits'' the original image.

In homotopic reconstructions, some reconstructible zones surrounded by nonreconstructible zones may never been reached. These zones correspond to holes in the object. If these holes are to be recovered, a nonhomotopic reconstruction operator must be used. The definition of this operator is the same as the definition of the reconstruction operator, except that we do not impose that the point x be simple (see Def. 12). The results given by this operator are represented Figs. 11(f) and 12(f). Some light and disconnected areas are recovered.

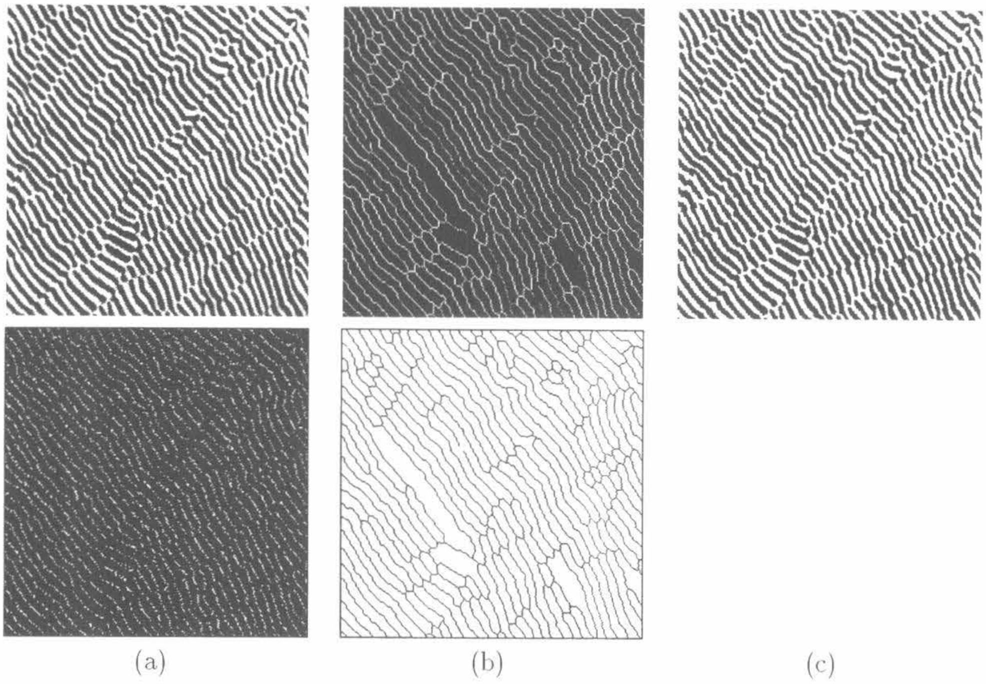

Fig. 10 Illustration of reconstruction: (a) original image; (b) lower leveling; and (c) reconstruction.

9 Conclusion

We have introduced a cross-section topology and some basic transformations relative to that topology, the homotopic and leveling transformations. Based on these notions, we have proposed a segmentation chain for grayscale images which consists in the three following steps:

Step 1 is the leveling transformation, which preserves some basic topological characteristics of the regions to be segmented. For example, it preserves the connectedness of minima. It also provides informations about the minima, which can be considered as neighbors, as well as the altitude of the pass, which separates two neighboring minima.

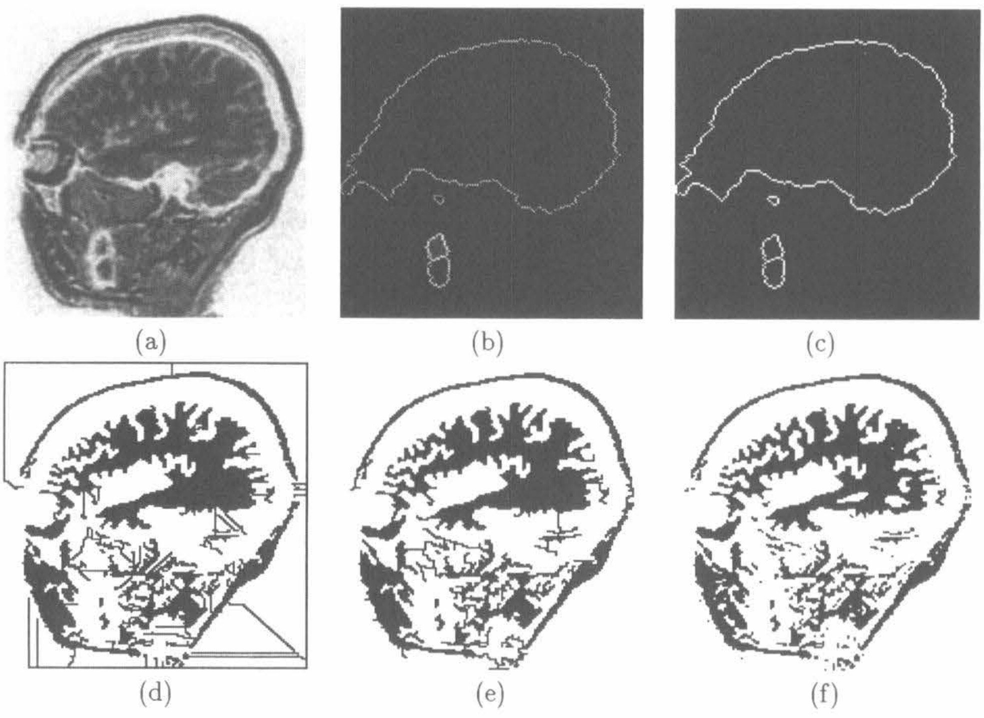

Fig. 11 Illustration of reconstruction: (a) original image; (b) regularization; (c) upper set; (d) homotopic reconstruction; (e) hierarchical reconstruction, and (f) nonhomotopic reconstruction.

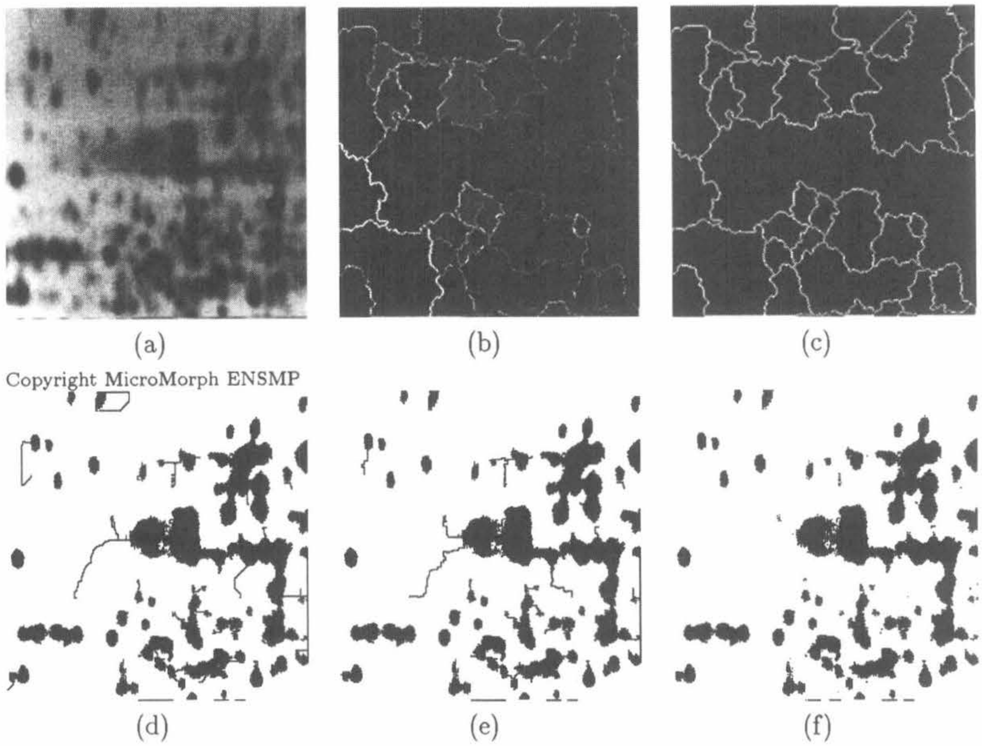

Fig. 12 Illustration of reconstruction: (a) original image; (b) regularization; (c) upper set, (d) homotopic reconstruction; (e) hierarchical reconstruction; and (f) nonhomotopic reconstruction.

The second step consists of two regularization transformations. They break the topology to eliminate nonsignificant regions. The upper regularization eliminates nonsignificant minima, while the lower regularization transformation eliminates nonsignificant upper regions.

The last step is the reconstruction transformation, which enables us to recover the shape of the original regions.

This approach to segmentation is new. It provides significant results even when the initial image has bad contrast. Note that the presented images have been segmented using only the preceding four basic operators; no enhancement or preprocessing step has been used. Furthermore, these operators do not require the tuning of any parameter.

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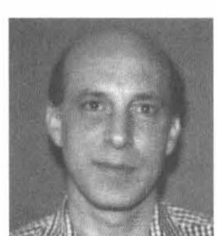
References

1. T. Y. Kong and A. Rosenfeld, "Digital topology: introduction and survey," *Comput. Vis. Graph. Image Process.* **48**, 357–393 (1989).
2. A. Rosenfeld, "Fuzzy digital topology," *Inf. Control* **40**, 76–87 (1979).
3. A. Rosenfeld, "On connectivity properties of grayscale pictures," *Pattern Recog.* **16**, 47–50 (1983).
4. S. Beucher and F. Meyer, "The morphological approach to segmentation: the watershed transformation," Chap. 12 in *Mathematical Morphology in Image Processing*, Dougherty, Ed., pp. 433–481, Marcel Dekker, New York (1993).
5. C. Dyer and A. Rosenfeld, "Thinning algorithms for gray-scale pictures," *IEEE Trans. Pattern Anal. Mach. Intel.* **1**, 88–89 (1979).
6. J. Serra, *Image Analysis and Mathematical Morphology*, Academic Press, London (1982).
7. S. Beucher, "Segmentation d'images et morphologie mathématique," PhD Thesis, Ecole des Mines, Paris (1990).
8. M. Grimaud, "A new measure of contrast: the dynamics," in *Image Algebra and Morphological Processing III, Proc. SPIE* **1769**, 292–305 (1992).
9. F. Meyer and S. Beucher, "Morphological segmentation," *J. Vis. Commun. Image Process.* **1**(1), 21–46 (1990).
10. L. Vincent and B. Masters, "Morphological processing and network analysis of cornea endothelial cell images," in *Image Algebra and Morphological Image Processing III, Proc. SPIE* **1769**, 212–225 (1992).
11. L. Vincent, "Algorithmes morphologiques à base de files d'attente et de lacets. Extension aux graphes," PhD Thesis, Ecole des Mines, Paris (1990).
12. J. P. Cocquerez and S. Philipp, *Analyse d'images: filtrage et segmentation*, Masson Ed., Paris (1995).
13. G. Bertrand, "Simple points, topological numbers and geodesic neighborhoods in cubic grids," *Pattern Recog. Lett.* **15**, 1003–1011 (1994).
14. F. Meyer, "Skeletons and perceptual graphs," *Signal Process.* **16**, 335–363 (1989).
15. F. Meyer, "Un algorithme optimal de ligne de partage des eaux," in *Proc. 8th Conf. Rec. des Formes et Int. Art.*, Vol. 2, pp. 847–859, AFCET Ed., Lyon (1992).
16. J. C. Everat and G. Bertrand, "New topological operators for segmentation," in *Proc. IEEE Int. Conf. on Image*, Vol. III/III, pp. 45–48 (1996).



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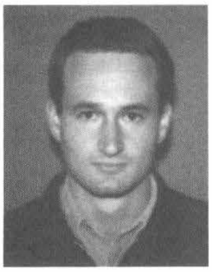

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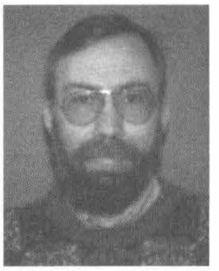

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